

SPECTRAL PROPERTIES OF A CLASS OF REFLECTIONLESS SCHRÖDINGER OPERATORS

FRITZ GESZTESY AND PETER YUDITSKII

ABSTRACT. We prove that one-dimensional reflectionless Schrödinger operators with spectrum a homogeneous set in the sense of Carleson, belonging to the class introduced by Sodin and Yuditskii, have purely absolutely continuous spectra. This class includes all earlier examples of reflectionless almost periodic Schrödinger operators.

In addition, we construct examples of reflectionless Schrödinger operators with more general types of spectra, given by the complement of a Denjoy–Widom-type domain in \mathbb{C} , which exhibit a singular component.

1. INTRODUCTION

In this paper we consider a certain class of reflectionless self-adjoint Schrödinger operators $H = -d^2/dx^2 + V$ in $L^2(\mathbb{R}; dx)$ and study the spectral properties of its members. In particular, assuming that the spectrum $\sigma(H)$ of H is given by a homogeneous set \mathcal{E} in the sense of Carleson, we prove that H has purely absolutely continuous spectrum. Subsequently, we consider reflectionless Schrödinger operators with more general types of spectra given by the complement of a Denjoy–Widom-type domain in \mathbb{C} and construct a class of examples that exhibits a singular component in its spectrum.

To put our results in proper perspective we first recall that a *homogeneous set* $\mathcal{E} \subset \mathbb{R}$, bounded from below, is of the type (cf. Carleson [9])

$$\mathcal{E} = [E_0, \infty) \setminus \bigcup_{j \in J} (a_j, b_j), \quad J \subseteq \mathbb{N}, \quad (1.1)$$

for some $E_0 \in \mathbb{R}$ and $a_j < b_j$, where $(a_j, b_j) \cap (a_{j'}, b_{j'}) = \emptyset$, $j, j' \in J$, $j \neq j'$ such that the following condition is fulfilled:

$$\begin{aligned} &\text{There exists an } \varepsilon > 0 \text{ such that for all } \lambda \in \mathcal{E} \\ &\text{and all } \delta > 0, \quad |\mathcal{E} \cap (\lambda - \delta, \lambda + \delta)| \geq \varepsilon \delta. \end{aligned} \quad (1.2)$$

We will also assume that \mathcal{E} has finite gap length, that is, $\sum_{j \in J} (b_j - a_j) < \infty$ is supposed to hold.

Reflectionless (self-adjoint) Schrödinger operators H in $L^2(\mathbb{R}; dx)$ can be characterized by the fact that for all $x \in \mathbb{R}$ and for a.e. $\lambda \in \sigma_{\text{ess}}(H)$, the diagonal Green's function of H has purely imaginary normal boundary values,

$$G(\lambda + i0, x, x) \in i\mathbb{R}. \quad (1.3)$$

Date: February 2, 2008.

2000 *Mathematics Subject Classification.* Primary 34B20, 34L05, 34L40; Secondary 34B24, 34B27, 47A10.

Based upon work partially supported by the US National Science Foundation under Grant No. DMS-0405526 and the Austrian Science Fund FWF, project number: P16390-N04.

Here $\sigma_{\text{ess}}(H)$ denotes the essential spectrum of H (we assume $\sigma_{\text{ess}}(H) \neq \emptyset$) and

$$G(z, x, x') = (H - zI)^{-1}(x, x'), \quad z \in \mathbb{C} \setminus \sigma(H), \quad (1.4)$$

denotes the integral kernel of the resolvent of H .

$H_0 = -d^2/dx^2$ and the N -soliton potentials V_N , $N \in \mathbb{N}$, that is, exponentially decreasing solutions in $C^\infty(\mathbb{R})$ of some (and hence infinitely many) equations of the stationary Korteweg–de Vries (KdV) hierarchy, yield well-known examples of reflectionless Schrödinger operators $H_N = -d^2/dx^2 + V_N$. However, since such operators have $N \in \mathbb{N}$ strictly negative eigenvalues, the spectrum of H_N , $N \in \mathbb{N}$, is not a homogeneous set. The prototype of reflectionless Schrödinger operators with a homogeneous spectrum are in fact the set of periodic Schrödinger operators. Indeed, if V_a is periodic with some period $a > 0$, that is, $V_a(x + a) = V_a(x)$ for a.e. $x \in \mathbb{R}$, then standard Floquet theoretic considerations show that the spectrum of $H_a = -d^2/dx^2 + V_a$ is a countable union of compact intervals (which may degenerate into a union of finitely-many compact intervals and a half-line) and hence a homogeneous set, and at the same time, the diagonal Green's function of H_a is purely imaginary for every point in the open interior of $\sigma(H_a)$. More generally, also certain classes of quasi-periodic and almost periodic potentials give rise to reflectionless Schrödinger operators with homogeneous spectra. The prime example of such quasi-periodic potentials is represented by the class of real-valued bounded algebro-geometric KdV potentials corresponding to an underlying (compact) hyperelliptic Riemann surface (see, e.g., [6, Ch. 3], [20], [26, Ch. 1], [37], [52, Chs. 8, 10], [55, Ch. 4], [58, Ch. II] and the literature cited therein). More general classes of almost periodic and reflectionless Schrödinger operators were studied by Avron and Simon [5], Carmona and Lacroix [10, Ch. VII], Chulaevskii [11], Craig [15], Deift and Simon [17], Egorova [22], Johnson [36], Johnson and Moser [38], Kotani [44]–[46], Kotani and Krishna [47], Levitan [48]–[51], [52, Chs. 9, 11], Levitan and Savin [54], Moser [56], Pastur and Figotin [59, Chs. V, VII], Pastur and Tkachenko [60], and more recently, by Sodin and Yuditskii [72]–[74], the starting point of our investigation.

In Section 2 we introduce the class $\text{SY}(\mathcal{E})$ of potentials associated with one-dimensional Schrödinger operators $H = -d^2/dx^2 + V$, whose spectra coincide with the homogeneous set \mathcal{E} . The class $\text{SY}(\mathcal{E})$ is possibly a slight generalization of the class of potentials studied by Sodin and Yuditskii [73], [74] in the sense that we do not assume continuity of the potential V from the outset. For precise details on $\text{SY}(\mathcal{E})$ we refer to Definition 2.3. Here we just mention that $V \in \text{SY}(\mathcal{E})$ if $\sigma(H) = \mathcal{E}$ and the half-line Weyl–Titchmarsh functions $m_\pm(z, x_0)$ (associated with the restriction of H to the half-lines $(x_0, \pm\infty)$) satisfy a certain pseudo-analytic continuation property across \mathcal{E} . We then derive the essential boundedness of such potentials using the trace formula proved in [28],

$$\text{SY}(\mathcal{E}) \subset L^\infty(\mathbb{R}; dx), \quad (1.5)$$

and set the stage for the remainder of this paper by showing that the corresponding Schrödinger operators are reflectionless.

In Section we prove by elementary methods that the absolutely continuous spectrum of H with $V \in \text{SY}(\mathcal{E})$ coincides with the prescribed homogeneous set \mathcal{E} ,

$$\sigma_{\text{ac}}(H) = \mathcal{E}, \quad (1.6)$$

and that H has uniform multiplicity equal to two. To obtain this result we exploit the notion of essential supports of measures and the essential closure of sets, a topic summarized in Appendix A.

In Section 3 we prove the absence of a singular component in the spectrum of H and together with (1.6) this then implies that

$$\sigma(H) = \sigma_{\text{ac}}(H) = \mathcal{E}, \quad \sigma_{\text{sc}}(H) = \sigma_{\text{pp}}(H) = \emptyset, \quad (1.7)$$

our principal result for potentials V in the class $\text{SY}(\mathcal{E})$. The proof of $\sigma_{\text{sc}}(H) = \sigma_{\text{pp}}(H) = \emptyset$ is more involved and requires some results from the theory of Hardy spaces, most notably, the space $H^1(\mathcal{E})$. The necessary prerequisites for this topic are summarized in Appendix B.

Finally, in Section 4 we construct a class of reflectionless Schrödinger operators with more general types of spectra given by the complement of a Denjoy–Widom-type domain in \mathbb{C} . In particular, we construct a class of examples which exhibits a particular accumulation point of spectral bands and gaps. As a result, H acquires an eigenvalue and hence a singular component in its spectrum.

Throughout the bulk of this paper we repeatedly exploit properties of Herglotz functions and their exponential representations as well as certain elements of Weyl–Titchmarsh and spectral multiplicity theory for self-adjoint Schrödinger operators on half-lines and on \mathbb{R} . A nutshell-type treatment of both topics is presented in Appendix B.

While we focus in this paper on one-dimensional Schrödinger operators, we emphasize that all methods employed apply to Jacobi, Dirac, and CMV-type operators and yield an analogous set of results.

2. THE SODIN–YUDITSKII CLASS OF REFLECTIONLESS POTENTIALS

In this section we describe the Sodin–Yuditskii class $\text{SY}(\mathcal{E})$ of reflectionless potentials associated with the homogeneous set \mathcal{E} bounded from below and recall some of the basic properties of one-dimensional Schrödinger operators with such potentials.

Before we analyze the class $\text{SY}(\mathcal{E})$ in some detail, we start with some general considerations of one-dimensional Schrödinger operators. Let

$$V \in L^1_{\text{loc}}(\mathbb{R}; dx), \quad V \text{ real-valued}, \quad (2.1)$$

and assume that the differential expression

$$L = -d^2/dx^2 + V(x), \quad x \in \mathbb{R} \quad (2.2)$$

is in the limit point case at $+\infty$ and $-\infty$. We denote by H the corresponding self-adjoint realization of L in $L^2(\mathbb{R}; dx)$ given by

$$\begin{aligned} Hf &= Lf, \\ f \in \text{dom}(H) &= \{g \in L^2(\mathbb{R}; dx) \mid g, g' \in AC_{\text{loc}}(\mathbb{R}); Lg \in L^2(\mathbb{R}; dx)\}. \end{aligned} \quad (2.3)$$

Let $g(z, \cdot)$ denote the diagonal Green’s function of H , that is,

$$g(z, x) = G(z, x, x), \quad G(z, x, x') = (H - zI)^{-1}(x, x'), \quad z \in \mathbb{C} \setminus \sigma(H), \quad x, x' \in \mathbb{R}. \quad (2.4)$$

Since for each $x \in \mathbb{R}$, $g(\cdot, x)$ is a Herglotz function (i.e., it maps the open complex upper half-plane analytically to itself),

$$\xi(\lambda, x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \text{Im}[\ln(g(\lambda + i\varepsilon, x))] \quad \text{for a.e. } \lambda \in \mathbb{R} \quad (2.5)$$

is well-defined for each $x \in \mathbb{R}$. In particular, for all $x \in \mathbb{R}$,

$$0 \leq \xi(\lambda, x) \leq 1 \text{ for a.e. } \lambda \in \mathbb{R}. \quad (2.6)$$

In the following we will frequently use the convenient abbreviation

$$h(\lambda_0 + i0) = \lim_{\varepsilon \downarrow 0} h(\lambda_0 + i\varepsilon), \quad \lambda_0 \in \mathbb{R}, \quad (2.7)$$

whenever the limit in (2.7) is well-defined and hence (2.5) can then be written as $\xi(\lambda, x) = (1/\pi)\text{Arg}(g(\lambda + i0, x))$. Moreover, we will use the convention that whenever the phrase a.e. is used without further qualification, it always refers to Lebesgue measure on \mathbb{R} .

Associated with H in $L^2(\mathbb{R}; dx)$ we also introduce the two half-line Schrödinger operators H_{\pm, x_0} in $L^2([x_0, \pm\infty); dx)$ with Dirichlet boundary conditions at the finite endpoint $x_0 \in \mathbb{R}$,

$$H_{\pm, x_0} f = Lf,$$

$$f \in \text{dom}(H_{\pm, x_0}) = \{g \in L^2([x_0, \pm\infty); dx) \mid g, g' \in AC([x_0, x_0 \pm R]) \text{ for all } R > 0; \\ \lim_{\varepsilon \downarrow 0} g(x_0 \pm \varepsilon) = 0; Lg \in L^2([x_0, \pm\infty); dx)\}. \quad (2.8)$$

The *half-line Weyl–Titchmarsh functions* associated with H_{\pm, x_0} (to be discussed in (2.33)–(2.35)), are denoted by $m_{\pm}(z, x_0)$, $z \in \mathbb{C} \setminus \sigma(H_{\pm, x_0})$.

Next, we recall the trace formula first proved in [28] and slightly refined in [14] (see also [67] for an interesting extension).

Theorem 2.1. *Let H be the self-adjoint Schrödinger operator defined in (2.3) and assume in addition that H is bounded from below, $E_0 = \inf(\sigma(H)) > -\infty$. Then,*

$$V(x) = E_0 + \lim_{z \rightarrow i\infty} \int_{E_0}^{\infty} d\lambda z^2 (\lambda - z)^{-2} [1 - 2\xi(\lambda, x)] \text{ for a.e. } x \in \mathbb{R}. \quad (2.9)$$

Proof. We briefly sketch the main arguments. The exponential Herglotz representation of $g(z, x)$ yields (cf. (2.19), (B.21), (B.23))

$$g(z, x) = [m_-(z, x) - m_+(z, x)]^{-1} \quad (2.10)$$

$$= \exp \left[c(x) + \int_{\mathbb{R}} d\lambda \xi(\lambda, x) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \right], \quad (2.11)$$

$$c(x) = \text{Re}[\ln(g(i, x))], \quad z \in \mathbb{C} \setminus \sigma(H), \quad x \in \mathbb{R}$$

and hence by (2.6),

$$\frac{d}{dz} \ln[g(z, x)] = \int_{E_0}^{\infty} d\lambda (\lambda - z)^{-2} \xi(\lambda, x), \quad z \in \mathbb{C} \setminus \sigma(H), \quad x \in \mathbb{R}. \quad (2.12)$$

Next, we denote by \mathcal{L}_V the set of Lebesgue points of V , that is,

$$\mathcal{L}_V = \left\{ x \in \mathbb{R} \mid \int_{-\varepsilon}^{\varepsilon} dx' |V(x + x') - V(x)| \underset{\varepsilon \downarrow 0}{=} o(\varepsilon) \text{ as } \varepsilon \downarrow 0 \right\}, \quad (2.13)$$

and we suppose that $x \in \mathcal{L}_V$. Let $C_{\varepsilon} \subset \mathbb{C}_+$ be the sector along the positive imaginary axis with vertex at zero and opening angle ε with $0 < \varepsilon < \pi/2$. Then, as proved in [12, Theorem 4.8], $m_{\pm}(z, x)$ has an asymptotic expansion of the form ($\text{Im}(z^{1/2}) \geq 0$, $z \in \mathbb{C}$)

$$m_{\pm}(z, x) \underset{\substack{|z| \rightarrow \infty \\ z \in C_{\varepsilon}}}{=} \pm i z^{1/2} \mp (i/2)V(x)z^{-1/2} + o(|z|^{-1/2}) \quad (2.14)$$

as $|z| \rightarrow \infty$. By (2.10), $g(z, x)$ has an asymptotic expansion in C_ε of the form

$$g(z, x) \underset{\substack{|z| \rightarrow \infty \\ z \in C_\varepsilon}}{=} (i/2)z^{-1/2} + (i/4)V(x)z^{-3/2} + o(|z|^{-3/2}) \quad (2.15)$$

as $|z| \rightarrow \infty$. Differentiation of (2.15) with respect to z then yields

$$-\frac{d}{dz} \ln[g(z, x)] \underset{z \rightarrow i\infty}{=} \frac{1}{2}z^{-1} + \frac{1}{2}V(x)z^{-2} + o(|z|^{-2}). \quad (2.16)$$

Thus,

$$\begin{aligned} -\frac{d}{dz} \ln[g(z, x)] &= \frac{1}{2}(z - E_0)^{-1} + \frac{1}{2} \int_{E_0}^{\infty} d\lambda (\lambda - z)^{-2} [1 - 2\xi(\lambda, x)] \\ &\underset{z \rightarrow i\infty}{=} \frac{1}{2}z^{-1} + \frac{1}{2}V(x)z^{-2} + o(|z|^{-2}) \end{aligned} \quad (2.17)$$

proves (2.9) for $x \in \mathcal{L}_V$. Since $\mathbb{R} \setminus \mathcal{L}_V$ has Lebesgue measure zero, (2.9) is proved. \square

For subsequent purpose we also note the universal asymptotic z -behavior of $m_\pm(z, x)$ and $g(z, x)$ valid for all $x \in \mathbb{R}$,

$$m_\pm(z, x) \underset{\substack{|z| \rightarrow \infty \\ z \in C_\varepsilon}}{=} \pm iz^{1/2} [1 + o(1)], \quad (2.18)$$

$$g(z, x) \underset{\substack{|z| \rightarrow \infty \\ z \in C_\varepsilon}}{=} (i/2)z^{-1/2} [1 + o(1)]. \quad (2.19)$$

Since Schrödinger operators bounded from below play a special role in our considerations, we take a closer look at them next. Given $V \in L^1_{\text{loc}}(\mathbb{R}; dx)$, V real-valued, and $L = -d^2/dx^2 + V$ as in (2.1) and (2.2), we define the associated minimal Schrödinger operator H_{\min} in $L^2(\mathbb{R}; dx)$ by

$$\begin{aligned} H_{\min} f &= Lf, \\ f \in \text{dom}(H_{\min}) &= \{g \in L^2(\mathbb{R}; dx) \mid g, g' \in AC_{\text{loc}}(\mathbb{R}); \text{supp}(g) \text{ compact}; \\ &\quad Lg \in L^2(\mathbb{R}; dx)\}. \end{aligned} \quad (2.20)$$

By a well-known result of Hartman [34] (see also Rellich [64], [25], and the literature cited in [13]), H_{\min} is essentially self-adjoint, or equivalently, the differential expression L is in the limit point case at $+\infty$ and $-\infty$ if H_{\min} is bounded from below. In this case, the operator $H = \overline{H_{\min}}$ (the operator closure of H_{\min}) is the unique self-adjoint extension of H_{\min} in $L^2(\mathbb{R}; dx)$ and it coincides with the maximally defined operator in (2.3).

Definition 2.2. Let $\mathcal{E} \subset \mathbb{R}$ be a closed set bounded from below which we may write as

$$\mathcal{E} = [E_0, \infty) \setminus \bigcup_{j \in J} (a_j, b_j), \quad J \subseteq \mathbb{N}, \quad (2.21)$$

for some $E_0 \in \mathbb{R}$ and $a_j < b_j$, where $(a_j, b_j) \cap (a_{j'}, b_{j'}) = \emptyset$, $j, j' \in J$, $j \neq j'$. Then \mathcal{E} is called *homogeneous* if

$$\begin{aligned} &\text{there exists an } \varepsilon > 0 \text{ such that for all } \lambda \in \mathcal{E} \\ &\text{and all } \delta > 0, \quad |\mathcal{E} \cap (\lambda - \delta, \lambda + \delta)| \geq \varepsilon \delta. \end{aligned} \quad (2.22)$$

Moreover, we say that \mathcal{E} is of *finite gap length* if

$$\sum_{j \in J} (b_j - a_j) < \infty. \quad (2.23)$$

Homogeneous sets were originally discussed by Carleson [9]; we also refer to [39], [61], and [81].

Next we introduce the Sodin–Yuditskii class $\text{SY}(\mathcal{E})$ of potentials associated with a homogeneous set \mathcal{E} of finite gap length.

Definition 2.3. Let $\mathcal{E} \subset \mathbb{R}$ be a homogeneous set of finite gap length of the form (2.21) and pick $x_0 \in \mathbb{R}$. Then $V \in L^1_{\text{loc}}(\mathbb{R}; dx)$ belongs to the Sodin–Yuditskii class $\text{SY}(\mathcal{E})$ associated with \mathcal{E} if

$$(i) \quad V \text{ is real-valued.} \quad (2.24)$$

$$(ii) \quad H_{\min} \text{ is bounded from below and its unique self-adjoint extension } H \text{ has spectrum } \sigma(H) = \mathcal{E}. \quad (2.25)$$

$$(iii) \quad \text{For a.e. } \lambda \in \mathcal{E}, \quad \lim_{\varepsilon \downarrow 0} m_+(\lambda + i\varepsilon, x_0) = \lim_{\varepsilon \downarrow 0} m_-(\lambda - i\varepsilon, x_0). \quad (2.26)$$

Next we will demonstrate that the Sodin–Yuditskii class $\text{SY}(\mathcal{E})$ is independent of the choice of $x_0 \in \mathbb{R}$ in (2.26) and that $\text{SY}(\mathcal{E}) \subset L^\infty(\mathbb{R}; dx)$. Moreover, potentials $V \in \text{SY}(\mathcal{E})$ are reflectionless in a sense that will be made precise below. These results have short proofs which we will present. Later in this section we will indicate that all potentials in $\text{SY}(\mathcal{E})$ are actually continuous on \mathbb{R} and uniformly (i.e., Bohr) almost periodic. Since the latter result requires quite different potential theoretic techniques we will quote them without proofs.

Theorem 2.4. *Let $V \in \text{SY}(\mathcal{E})$. Then, V is reflectionless in the sense that*

$$\text{for each } x \in \mathbb{R}, \quad \xi(\lambda, x) = 1/2 \text{ for a.e. } \lambda \in \mathcal{E}. \quad (2.27)$$

Moreover,

$$\text{SY}(\mathcal{E}) \subset L^\infty(\mathbb{R}; dx) \quad (2.28)$$

and $\text{SY}(\mathcal{E})$ is independent of the choice of $x_0 \in \mathbb{R}$ in Definition 2.3(iii).

Proof. We introduce the usual fundamental system of solutions $\phi(z, \cdot, x_0)$ and $\theta(z, \cdot, x_0)$, $z \in \mathbb{C}$, with respect to a fixed reference point $x_0 \in \mathbb{R}$, of solutions of

$$L\psi(z, x) = z\psi(z, x), \quad (z, x) \in \mathbb{C} \times \mathbb{R}, \quad (2.29)$$

satisfying the initial conditions at the point $x = x_0$,

$$\phi(z, x_0, x_0) = \theta'(z, x_0, x_0) = 0, \quad \phi'(z, x_0, x_0) = \theta(z, x_0, x_0) = 1. \quad (2.30)$$

Then for any fixed $x, x_0 \in \mathbb{R}$, $\phi(z, x, x_0)$ and $\theta(z, x, x_0)$ are entire with respect to z ,

$$W(\theta(z, \cdot, x_0), \phi(z, \cdot, x_0))(x) = 1, \quad z \in \mathbb{C}, \quad (2.31)$$

and

$$\overline{\phi(z, x, x_0)} = \phi(\bar{z}, x, x_0), \quad \overline{\theta(z, x, x_0)} = \theta(\bar{z}, x, x_0), \quad (z, x, x_0) \in \mathbb{C} \times \mathbb{R}^2. \quad (2.32)$$

Particularly important solutions of (2.29) are the so called *Weyl–Titchmarsh solutions* $\psi_\pm(z, \cdot, x_0)$, $z \in \mathbb{C} \setminus \sigma(H)$, uniquely characterized by

$$\psi_\pm(z, \cdot, x_0) \in L^2([x_0, \pm\infty); dx), \quad z \in \mathbb{C} \setminus \sigma(H), \quad \psi_\pm(z, x_0, x_0) = 1. \quad (2.33)$$

The normalization in (2.33) shows that $\psi_{\pm}(z, \cdot, x_0)$ is of the type

$$\psi_{\pm}(z, x, x_0) = \theta(z, x, x_0) + m_{\pm}(z, x_0)\phi(z, x, x_0), \quad z \in \mathbb{C} \setminus \sigma(H), \quad x \in \mathbb{R}, \quad (2.34)$$

with $m_{\pm}(z, x_0)$, the Weyl–Titchmarsh m -functions associated with the Dirichlet half-line Schrödinger operators H_{\pm, x_0} in (2.8). We recall that $\pm m_{\pm, x_0}$ are Herglotz functions and that

$$\overline{m_{\pm}(z, x_0)} = m_{\pm}(\bar{z}, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.35)$$

We also note that

$$g(z, x) = \frac{\psi_+(z, x, x_0)\psi_-(z, x, x_0)}{W(\psi_+(z, x, x_0), \psi_-(z, x, x_0))} = \frac{\psi_+(z, x, x_0)\psi_-(z, x, x_0)}{m_-(z, x_0) - m_+(z, x_0)}, \quad (2.36)$$

$$m_{\pm}(z, x) = \frac{\psi'_{\pm}(z, x, x_0)}{\psi_{\pm}(z, x, x_0)}, \quad z \in \mathbb{C} \setminus \mathcal{E}, \quad x \in \mathbb{R}. \quad (2.37)$$

By (2.26) and (2.35) one concludes that

$$m_+(\lambda + i0, x_0) = m_-(\lambda - i0, x_0) = \overline{m_-(\lambda + i0, x_0)} \text{ for a.e. } \lambda \in \mathcal{E} \quad (2.38)$$

and together with (2.32) (implying that $\phi(\lambda, x, x_0)$ and $\theta(\lambda, x, x_0)$ are real-valued for all $(\lambda, x, x_0) \in \mathbb{R}^3$) this proves that

$$\text{for each } x \in \mathbb{R}, \quad g(\lambda + i0, x) \text{ is purely imaginary for a.e. } \lambda \in \mathcal{E}. \quad (2.39)$$

By (2.5) this proves (2.27).

Since for each $x \in \mathbb{R}$, $g(z, x)$ is necessarily real-valued in spectral gaps $[E_0, \infty) \setminus \mathcal{E}$, $\xi(\lambda, x) \in \{0, 1\}$ for $\lambda \in (a_j, b_j)$, $j \in J$, whenever $g(\lambda, x) \neq 0$. Moreover, since

$$\frac{d}{dz}(H - z)^{-1} = (H - z)^{-2}, \quad z \in \mathbb{C} \setminus \sigma(H), \quad (2.40)$$

one has for fixed $x \in \mathbb{R}$,

$$\frac{d}{d\lambda}g(\lambda, x) = \int_{\mathbb{R}} dx' G(\lambda, x, x')^2 > 0, \quad \lambda \in (-\infty, E_0) \cup \bigcup_{j \in J} (a_j, b_j). \quad (2.41)$$

Thus, for fixed $x \in \mathbb{R}$, $g(\cdot, x)$ is strictly monotonically increasing on each interval (a_j, b_j) , $j \in J$ and hence one obtains the following behavior of $\xi(\cdot, x)$ on $[E_0, \infty) \setminus \mathcal{E}$:

$$\xi(\lambda, x) = \begin{cases} 1, & \lambda \in (a_j, \mu_j(x)) \\ 0, & \lambda \in (\mu_j(x), b_j) \end{cases} \text{ if } \mu_j(x) \in (a_j, b_j), \quad j \in J, \quad (2.42)$$

$$\xi(\lambda, x) = \begin{cases} 1, & \lambda \in (a_j, b_j) \text{ if } g(\lambda, x) < 0, \lambda \in (a_j, b_j), \quad j \in J, \\ 0, & \lambda \in (a_j, b_j) \text{ if } g(\lambda, x) > 0, \lambda \in (a_j, b_j), \quad j \in J, \end{cases} \quad (2.43)$$

where

$$g(\mu_j(x), x) = 0, \text{ whenever } \mu_j(x) \in (a_j, b_j), \quad j \in J. \quad (2.44)$$

For convenience we will also define

$$\mu_j(x) = \begin{cases} b_j & \text{if } g(\lambda, x) < 0, \lambda \in (a_j, b_j), \quad j \in J, \\ a_j & \text{if } g(\lambda, x) > 0, \lambda \in (a_j, b_j), \quad j \in J. \end{cases} \quad (2.45)$$

Insertion of (2.42) and (2.43) into the trace formula (2.9), and using the convention (2.45) then yields

$$V(x) = E_0 + \sum_{j \in J} [a_j + b_j - 2\mu_j(x)] \text{ for a.e. } x \in \mathbb{R}. \quad (2.46)$$

Absolute convergence of the trace formula in (2.46) is assured by the finite gap length condition (2.23) of \mathcal{E} . Since by definition,

$$\mu_j(x) \in [a_j, b_j], \quad j \in J, \quad (2.47)$$

condition (2.23) proves

$$|V(x)| \leq |E_0| + \sum_{j \in J} |b_j - a_j| < \infty \text{ for a.e. } x \in \mathbb{R}. \quad (2.48)$$

In particular, $V \in L^\infty(\mathbb{R}; dx)$, proving (2.28).

Since by (2.37),

$$m_\pm(z, x) = \frac{\psi'_\pm(z, x, x_0)}{\psi_\pm(z, x, x_0)} = \frac{\theta'(z, x, x_0) + m_\pm(z, x_0)\phi'(z, x, x_0)}{\theta(z, x, x_0) + m_\pm(z, x_0)\phi(z, x, x_0)}, \quad (2.49)$$

one infers that for each $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} m_+(\lambda + i\varepsilon, x) = \overline{\lim_{\varepsilon \downarrow 0} m_-(\lambda + i\varepsilon, x)} = \lim_{\varepsilon \downarrow 0} m_-(\lambda - i\varepsilon, x) \text{ for a.e. } \lambda \in \mathcal{E}, \quad (2.50)$$

since for all $(x, x_0) \in \mathbb{R}^2$, $\theta(z, x, x_0)$ and $\phi(z, x, x_0)$ are entire with respect to z and real-valued for all $z \in \mathbb{R}$ (cf. (2.32)). Thus, (2.26) holds for all $x_0 \in \mathbb{R}$ and Definition 2.3 (iii) is independent of the choice of $x_0 \in \mathbb{R}$. \square

We will also call a Schrödinger operator H reflectionless if its potential coefficient V is reflectionless since hardly any confusion can arise in this manner.

Incidentally, inserting (2.42) and (2.43) into (2.11) yields the absolutely convergent product expansion for the diagonal Green's function g ,

$$g(z, x) = \frac{i}{2(z - E_0)^{1/2}} \prod_{j \in J} \frac{[z - \mu_j(x)]}{[(z - a_j)(z - b_j)]^{1/2}}, \quad z \in \mathbb{C} \setminus \mathcal{E}, \quad x \in \mathbb{R}. \quad (2.51)$$

At first sight, our class $\text{SY}(\mathcal{E})$ of potentials V appears to be larger than the class studied by Sodin and Yuditskii in [73] (see also [74]), since we did not assume continuity of V from the outset. A careful examination of the proofs in Sodin and Yuditskii in [73] and [75], however, shows that the assumption of continuity of V is unnecessary and automatically implied by conditions (i)–(iii) in Definition 2.3. Moreover, all elements $V \in \text{SY}(\mathcal{E})$ are uniformly almost periodic.

In the following we denote by $\text{AP}(\mathbb{R})$ the set of uniformly (i.e., Bohr) almost periodic (and hence continuous) functions on \mathbb{R} .

Theorem 2.5. *In addition to the boundedness and reflectionless properties of the class $\text{SY}(\mathcal{E})$ described in Theorem 2.4 one has that*

$$\text{SY}(\mathcal{E}) \subset \text{AP}(\mathbb{R}). \quad (2.52)$$

In particular,

$$\text{SY}(\mathcal{E}) \subset C(\mathbb{R}) \quad (2.53)$$

and hence the class $\text{SY}(\mathcal{E})$ introduced in Definition 2.3 coincides with the one studied in [73].

While a proof of (2.52), combining a variety of results of [73] and [75], is beyond the scope of this paper, we nevertheless briefly sketch some of the steps involved, simultaneously filling a gap in the proof of the approximation theorem in Section

5 of [73]:

(i) We recall the notion of Dirichlet data $\mathcal{D}_j(x)$ and Dirichlet divisors $\mathcal{D}(x)$

$$\begin{aligned} \mathcal{D}_j(x) &= (\mu_j(x), \sigma_j(x)), \quad \mu_j(x) \in [a_j, b_j], \quad \sigma_j(x) \in \{1, -1\}, \quad j \in J, \\ \mathcal{D}(x) &= \{\mathcal{D}_j(x)\}_{j \in J}, \quad x \in \mathbb{R}, \end{aligned} \quad (2.54)$$

where $\sigma_j(x) = 1$ (resp., $\sigma_j(x) = -1$) if $\mu_j(x) \in (a_j, b_j)$ is a right (resp., left) Dirichlet eigenvalue associated with the half-line $[x, \infty)$ (resp., $(-\infty, x]$), that is, $\mu_j(x) \in (a_j, b_j)$ is a pole of $m_+(z, x)$ (resp., $m_-(z, x)$). If $\mu_j(x)$ coincides with one of the endpoints a_j or b_j one identifies $(\mu_j(x), 1)$ and $(\mu_j(x), -1)$ and simply writes $(\mu_j(x))$. To avoid that $\sigma_j(x)$ formally becomes undefined whenever $\mu_j(x)$ hits an endpoint a_j or b_j , one can follow [15] and change variables from $(\mu_j(x), \sigma_j(x))$ to $\varphi(x)$ according to

$$\begin{aligned} \mu_j(x) &= [(a_j + b_j) + (b_j - a_j) \cos(\varphi_j(x))]/2, \\ \mu_j(x) &\in [a_j, b_j], \quad 0 < \varphi_j(x) < \pi \text{ for } \sigma_j(x) = 1, \quad \pi < \varphi_j(x) < 2\pi \text{ for } \sigma_j(x) = -1, \end{aligned} \quad (2.55)$$

without ambiguity as $\mu_j(x)$ equals a_j or b_j . As x varies in \mathbb{R} , $\varphi_j(x)$ traces out $[0, 2\pi]$ and hence we may identify the corresponding motion of $\mathcal{D}_j(x)$ with a circle \mathbb{T}_j and that of $\mathcal{D}(x)$ with a torus $\mathcal{D}(\mathcal{E})$ of dimension equal to the cardinality $|J|$ of the index set J . The torus $\mathcal{D}(\mathcal{E})$ will be equipped with the product topology which in turn may be generated by the norm

$$\|\varphi\| = \sum_{j \in J} 2^{-j} |\varphi_j|_{\text{mod}(2\pi)}, \quad \varphi = \{\varphi_j\}_{j \in J} \in \mathcal{D}(\mathcal{E}), \quad \varphi_j \in \mathbb{T}_j, \quad j \in J. \quad (2.56)$$

(ii) Fix $x_0 \in \mathbb{R}$ and a divisor $\mathcal{D}(x_0) \in \mathcal{D}(\mathcal{E})$. Then $\mathcal{D}(x_0)$ uniquely determines a potential $V \in \text{SY}(\mathcal{E})$ a.e. on \mathbb{R} . The proof of this fact utilizes the Borg–Marchenko uniqueness result for Schrödinger operators on the half-lines $(-\infty, x_0]$ and $[x_0, \infty)$ (to the effect that $m_{\pm}(z, x_0)$ uniquely determine the potential V a.e. on $[x_0, \pm\infty)$), the Abel map defined in terms of the harmonic measure of subsets of \mathcal{E} at $z \in \mathbb{C} \setminus \mathcal{E}$ with respect to $\mathbb{C} \setminus \mathcal{E}$, and the absence of singular inner factors of $\tilde{m}_{\pm}(z, x_0)$ and $\tilde{m}_{+}(z, x_0)\tilde{m}_{-}(z, x_0)$, where $\tilde{m}_{\pm}(z, x_0) = m_{\pm}(z, x_0) - m_{\pm}(-1, x_0)$.

(iii) As a final step one proves the finite-band approximation theorem. In this context we note, that the proof of the corresponding claim in [73, Sect. 5] is valid only in the sense of convergence on all compact subsets of the real axis, even though [73] contains the erroneous statement that the proof presented yields uniform convergence on \mathbb{R} . Indeed (following the notation in [73] in this paragraph with the exception that we denote E in [73] consistently by \mathcal{E}), the last displayed formula in [73, p. 652] only claims pointwise convergence for all $t \in \mathbb{R}$ whereas the first formula in [73, p. 653] already claims uniform convergence. (E.g., the sequence $\{e^{i(t/n)}\}_{n \in \mathbb{N}}$ pointwise approximates the constant function $f(t) = 1$, $t \in \mathbb{R}$, as $n \uparrow \infty$, but not uniformly on the whole real axis). However, this error is readily fixed: One only has to change the choice of approximation of the spectrum \mathcal{E} by finite band sets $\{\mathcal{E}^{(N)}\}_{N \in \mathbb{N}}$. To have uniform convergence as $N \uparrow \infty$ with respect to $t \in \mathbb{R}$ in the last displayed formula on p. 652 in [73], one has to keep fixed the initial frequencies of the approximating sets $\delta(\mathcal{E}^{(N)})$: That is, the initial coordinates of the vector $\delta(\mathcal{E}^{(N)})$ should *coincide* with the corresponding coordinates of $\delta(\mathcal{E})$. Then $A^{(N)}(D^{(N)}) + \delta(\mathcal{E}^{(N)})t$ indeed converges to $A(D) + \delta(\mathcal{E})t$ as $N \uparrow \infty$, uniformly in $t \in \mathbb{R}$, with respect to the chosen topology in $\pi^*(\mathcal{E})$.

This choice of approximation is well-known in spectral theory. The spectral set \mathcal{E} is usually described in terms of a conformal map on a comb-like domain. One defines

$$\Pi = \{w = u + iv \in \mathbb{C} \mid u > 0, v > 0\} \setminus \bigcup_{j \in J} \{w = u_j + iv \in \mathbb{C} \mid 0 < v \leq h_j\} \quad (2.57)$$

with $\sum_{j \in J} h_j < \infty$. Let $w = \Theta(\cdot)$ be a conformal map of the upper half-plane \mathbb{C}_+ onto Π with the normalizations $E_0 \mapsto 0$, $-\infty \mapsto i\infty$, and in addition,

$$\Theta(z) \underset{z \rightarrow -\infty}{=} z^{1/2}[1 + o(1)]. \quad (2.58)$$

Then $\mathcal{E} = \Theta^{-1}(\mathbb{R}_+)$, pre-images of slits form the system of intervals (a_j, b_j) , which represent the spectral gaps, and the set $\{u_j\}_{j \in J}$ forms the collection of frequencies of the corresponding almost periodic potential.

Let $\{J_N\}_{N \in \mathbb{N}}$ be an exhaustion of J by finite subsets:

$$\cdots J_N \subset J_{N+1} \subset \cdots \subset J. \quad (2.59)$$

Next, define

$$\Pi_N = \{w = u + iv \in \mathbb{C} \mid u > 0, v > 0\} \setminus \bigcup_{j \in J_N} \{w = u_j + iv \in \mathbb{C} \mid 0 < v \leq h_j\}, \quad (2.60)$$

introduce Θ_N as the conformal map onto Π_N under the same normalizations, and let $\mathcal{E}^{(N)} = \Theta_N^{-1}(\mathbb{R}_+)$, $N \in \mathbb{N}$.

Finally we define the map ϕ_N (a conformal map onto its image) by the diagram

$$\begin{array}{ccc} \mathbb{C}_+ & \xrightarrow{\Theta} & \Pi \\ \downarrow \phi_N & & \downarrow \text{id} \\ \mathbb{C}_+ & \xrightarrow{\Theta_N} & \Pi_N \end{array} \quad (2.61)$$

where id denotes the identity map.

Let $V \in \text{SY}(\mathcal{E})$ with uniquely associated divisor $\mathcal{D}(x_0)$ for some fixed $x_0 \in \mathbb{R}$. Associated with $\mathcal{D}(x_0) = \{\mathcal{D}_j(x_0)\}_{j \in J}$ are the “projections”

$$\mathcal{D}^{(N)}(x_0) = \{\phi_N(\mathcal{D}_j(x_0))\}_{j=1}^N, \quad \phi_N((\mu_j, \sigma_j)) = ((\phi_N(\mu_j), \sigma_j)), \quad 1 \leq j \leq N, \quad (2.62)$$

associated with $\mathcal{E}^{(N)}$. Applying the classical Jacobi inversion formula to the compact hyperelliptic Riemann surface associated with $\mathcal{E}^{(N)}$, one uniquely determines $\mathcal{D}^{(N)}(x)$ for all $x \in \mathbb{R}$ from the initial data $\mathcal{D}^{(N)}(x_0)$ and then constructs an algebro-geometric finite-band potential $V^{(N)}$ by using the trace formula

$$V^{(N)}(x) = E_0 + \sum_{j=1}^N [\phi_N(a_j) + \phi_N(b_j) - 2\phi_N(\mu_j(x))], \quad x \in \mathbb{R}. \quad (2.63)$$

In addition, one proves that the set of algebro-geometric finite-band potentials $V^{(N)}$ is precompact in the topology of uniform convergence on \mathbb{R} , and that each limit potential belongs to the class $\text{SY}(\mathcal{E})$. In particular, upon embedding $\mathcal{D}^{(N)}(x_0)$ into $\mathcal{D}(\mathcal{E})$ by introducing

$$\tilde{\mathcal{D}}^{(N)}(x_0) = \{(\mu_j(x_0), \sigma_j(x_0))\}_{j=1}^N \cup \{(b_k)\}_{k=N+1}^\infty \in \mathcal{D}(\mathcal{E}), \quad (2.64)$$

there exists a subsequence $\tilde{\mathcal{D}}^{(N_\ell)}(x_0)$, $\ell \in \mathbb{N}$, which converges to $\mathcal{D}(x_0)$ as $\ell \uparrow \infty$. Using the linearization property of the Abel map, one can then prove that actually

$\tilde{\mathcal{D}}^{(N_\ell)}(x)$ converges to $\mathcal{D}(x)$ as $\ell \uparrow \infty$ uniformly with respect to $x \in \mathbb{R}$. By the trace formula (2.63), the corresponding potentials $V^{(N_\ell)}$ converges to V as $\ell \uparrow \infty$ uniformly on \mathbb{R} . Since $V^{(N_\ell)}$ are continuous and quasi-periodic, one thus concludes that V is continuous and almost periodic.

The actual proofs of the statements in items (i)–(iii) in [73] and [75] rely in part on potential theoretic techniques, most notably, the theory of character-automorphic Hardy spaces. However, due to the above correction in choosing the proper approximating sequence, one additional result is needed to ensure convergence of $V^{(N)}(x_0)$ (cf. (2.63)) to $V(x_0)$ for a fixed divisor $\mathcal{D}(x_0)$. (In the original paper [73], ϕ_N played the role of the identity map at this place and hence the corresponding claim was obvious.)

Lemma 2.6. *Let ϕ_N be defined by (2.61). Then*

$$\phi_N(b_j) - \phi_N(a_j) \geq b_j - a_j \text{ for all } j \in J_N. \quad (2.65)$$

On the other hand,

$$\sum_{j \in J_N} [\phi_N(b_j) - \phi_N(a_j)] \leq \sum_{j \in J} (b_j - a_j). \quad (2.66)$$

In particular, this implies the uniform estimate

$$\sum_{\{j \in J_N \mid j \geq N\}} [\phi_N(b_j) - \phi_N(a_j)] \leq \sum_{\{j \in J \mid j \geq N\}} (b_j - a_j) \quad (2.67)$$

with respect to $N \in \mathbb{N}$.

Proof. We introduce the intermediate domain

$$\begin{aligned} \Pi_N^\tau &= \{w = u + iv \in \mathbb{C} \mid u > 0, v > 0\} \\ &\setminus \bigcup_{j \in J_N \cup \{k \in J \mid u_k \geq \tau\}} \{w = u_j + iv \in \mathbb{C} \mid 0 < v \leq h_j\}, \end{aligned} \quad (2.68)$$

and assume that $\tau > 0$ is a sufficiently large parameter, $\tau \neq u_j$, $j \in J$, that later on will be sent to $+\infty$. The corresponding function $\phi_N^\tau(z) = (\Theta_N^\tau)^{-1}(\Theta(z))$ maps the upper half plane in the upper half plane and therefore possesses the representation

$$\phi_N^\tau(z) = z + C_N^\tau + \int_0^{x(\tau)} \frac{d\rho_N^\tau(x)}{x - z}, \quad (2.69)$$

where $d\rho_N^\tau$ is a nonnegative measure, $C_N^\tau \in \mathbb{R}$, and $x(\tau) = \Theta^{-1}(\tau)$. We note that for each interval (a, b) , which is free of charge of the measure $d\rho$, we have

$$\phi_N^\tau(b) - \phi_N^\tau(a) = b - a + \int_0^{x(\tau)} \frac{(b - a)d\rho_N^\tau(x)}{(x - b)(x - a)} \geq b - a. \quad (2.70)$$

That is, the map is expanding on (a, b) . In particular,

$$\phi_N^\tau(b_j) - \phi_N^\tau(a_j) \geq b_j - a_j, \quad j \in J_N. \quad (2.71)$$

On the other hand, the set

$$\mathcal{E}^\tau = [0, x(\tau)] \setminus \bigcup_{\{j \in J \mid u_j \leq \tau\}} (a_j, b_j) \quad (2.72)$$

is also free of charge of the measure $d\rho$, that is, the measure of the image of this set is also larger than the measure of the set \mathcal{E}^τ . Thus (taking into account that $\phi_N^\tau(0) = 0$),

$$\phi_N^\tau(x(\tau)) - \sum_{j \in J_N} (\phi_N^\tau(b_j) - \phi_N^\tau(a_j)) \geq x(\tau) - \sum_{\{j \in J \mid u_j \leq \tau\}} (b_j - a_j), \quad (2.73)$$

and hence

$$\sum_{j \in J_N} [\phi_N^\tau(b_j) - \phi_N^\tau(a_j)] + x(\tau) - \phi_N^\tau(x(\tau)) \leq \sum_{\{j \in J \mid u_j \leq \tau\}} (b_j - a_j) \leq \sum_{j \in J} (b_j - a_j). \quad (2.74)$$

Using again the integral representation (2.69) one gets $x(\tau) \geq \phi_N^\tau(x(\tau))$ and therefore,

$$\sum_{j \in J_N} [\phi_N^\tau(b_j) - \phi_N^\tau(a_j)] \leq \sum_{j \in J} (b_j - a_j). \quad (2.75)$$

Keeping N fixed and sending τ to infinity we get (2.65) from (2.71), and (2.66) from (2.75) by Caratheodory's theorem [8], [33, Sect. II.5]. \square

Finally, we conclude this section with a converse to Theorem 2.4 due to Sodin and Yuditskii [73].

Theorem 2.7 ([73]). *Let $V \in L_{\text{loc}}^1(\mathbb{R}; dx)$ be real-valued and assume that H_{\min} is bounded from below and its unique self-adjoint operator H has spectrum \mathcal{E} , where \mathcal{E} is a homogeneous set of finite gap length of the form (2.21). In addition, suppose that V is reflectionless in the sense that*

$$\text{for each } x \in \mathbb{R}, \quad \xi(\lambda, x) = 1/2 \text{ for a.e. } \lambda \in \mathcal{E}. \quad (2.76)$$

Then,

$$V \in \text{SY}(\mathcal{E}) \quad (2.77)$$

and

$$\text{for all } x \in \mathbb{R} \text{ and for a.e. } \lambda \in \mathcal{E}, \quad \lim_{\varepsilon \downarrow 0} m_+(\lambda + i\varepsilon, x) = \lim_{\varepsilon \downarrow 0} m_-(\lambda - i\varepsilon, x). \quad (2.78)$$

Proof. In the Appendix of [73] it is proved that (2.76) implies that

$$\text{for a.e. } \lambda \in \mathcal{E}, \quad \lim_{\varepsilon \downarrow 0} m_+(\lambda + i\varepsilon, x_0) = \lim_{\varepsilon \downarrow 0} m_-(\lambda - i\varepsilon, x_0), \quad (2.79)$$

implying (2.77). Using (2.49), and noticing that $\theta(\lambda, x, x_0)$, $\theta'(\lambda, x, x_0)$, $\phi(\lambda, x, x_0)$, and $\phi'(\lambda, x, x_0)$ are all real-valued for $\lambda \in \mathcal{E}$, one concludes (2.78) (this argument has also been used to arrive at (2.50)). \square

In this section we shall prove that the set \mathcal{E} , the spectrum of a reflectionless Schrödinger operator H with potential V in the Sodin–Yuditskii class $\text{SY}(\mathcal{E})$, coincides with the absolutely continuous spectrum, $\sigma_{\text{ac}}(H)$, of H . In fact, we will provide two elementary proofs of this fact.

We start by recalling the following result on essential supports of the absolutely continuous spectrum of Schrödinger operators on the real line proven in [28] (see also [3, p. 383]). For completeness we will provide its short proof.

Let $V \in L_{\text{loc}}^1(\mathbb{R}; dx)$ be real-valued, assume that the differential expression $L = -d^2/dx^2 + V$ is in the limit point case at $+\infty$ and $-\infty$, and denote by H the corresponding self-adjoint realization of L in $L^2(\mathbb{R}; dx)$ introduced in (2.3). In addition, let $g(z, x)$, $z \in \mathbb{C} \setminus \sigma(H)$, $x \in \mathbb{R}$, and $\xi(\lambda, x)$ for a.e. $\lambda \in \mathbb{R}$, $x \in \mathbb{R}$, be

defined as in (2.4) and (2.5). By \overline{A}^e we denote the essential closure of the Lebesgue measurable set $A \subset \mathbb{R}$ (cf. Appendix A for details).

Theorem 2.8 ([28]). *For each $x \in \mathbb{R}$, the set*

$$\{\lambda \in \mathbb{R} \mid 0 < \xi(\lambda, x) < 1\} \quad (2.80)$$

is an essential support of the absolutely continuous spectrum, $\sigma_{\text{ac}}(H)$, of H . In particular, for each $x \in \mathbb{R}$,

$$\sigma_{\text{ac}}(H) = \overline{\{\lambda \in \mathbb{R} \mid 0 < \xi(\lambda, x) < 1\}}^e. \quad (2.81)$$

Proof. Pick $x \in \mathbb{R}$ and denote by $m_{\pm}(\cdot, x)$ the two half-line Weyl–Titchmarsh functions associated with the $H_{\pm, x}^D$ as defined in (2.8). Using (2.5) and (2.10), the following three sets coincide up to sets of Lebesgue measure zero:

$$(i) \quad \{\lambda \in \mathbb{R} \mid 0 < \xi(\lambda, x) < 1\}, \quad (2.82)$$

$$(ii) \quad \{\lambda \in \mathbb{R} \mid 0 < \text{Im}[g(\lambda + i0, x)] \text{ exists finitely}\}, \quad (2.83)$$

$$(iii) \quad S_{+, x} \cup S_{-, x}, \quad S_{\pm, x} = \{\lambda \in \mathbb{R} \mid 0 < \text{Im}[m_{\pm}(\lambda + i0, x)] \text{ exists finitely}\}. \quad (2.84)$$

By (B.49), $S_{\pm, x}$ are essential supports of $\sigma_{\text{ac}}(H_{\pm, x})$. But since H and $H_{-, x} \oplus H_{+, x}$ have the same absolutely continuous spectrum (their resolvents only differ by a rank-one perturbation), (2.82) represents an essential support of $\sigma_{\text{ac}}(H)$ proving the first claim concerning (2.80). Equation (2.81) is then clear from (A.19). \square

Next, let $\text{SY}(\mathcal{E})$ be the Sodin–Yuditskii class introduced in Definition 2.3. Given $V \in \text{SY}(\mathcal{E})$, we consider the Schrödinger operator H in $L^2(\mathbb{R}; dx)$ by

$$Hf = -f'' + Vf, \quad f \in \text{dom}(H) = H^{2,2}(\mathbb{R}) \quad (2.85)$$

with spectrum, $\sigma(H)$, of H given by

$$\sigma(H) = \mathcal{E}, \quad \mathcal{E} = [E_0, \infty) \setminus \bigcup_{j \in J} (a_j, b_j), \quad J \subseteq \mathbb{N}, \quad (2.86)$$

according to (2.21) and (2.25).

The principal result of this section then reads as follows.

Theorem 2.9. *Let $V \in \text{SY}(\mathcal{E})$ and denote by H the associated Schrödinger operator defined in (2.85). Then, the absolutely continuous spectrum of H coincides with \mathcal{E} ,*

$$\sigma_{\text{ac}}(H) = \mathcal{E}. \quad (2.87)$$

Moreover, $\sigma_{\text{ac}}(H)$ has uniform multiplicity equal to two.

Proof. Since V is reflectionless by Theorem 2.4, one has

$$\text{for each } x \in \mathbb{R}, \quad \xi(\lambda, x) = (1/\pi) \text{Im}[\ln(g(\lambda + i0, x))] = 1/2 \text{ for a.e. } \lambda \in \mathcal{E}. \quad (2.88)$$

By (2.81), this implies

$$\sigma_{\text{ac}}(H) = \overline{\{\lambda \in \mathbb{R} \mid 0 < \xi(\lambda, x_0) < 1\}}^e \supseteq \overline{\mathcal{E}}^e \quad (2.89)$$

for some $x_0 \in \mathbb{R}$. By the definition of homogeneity of \mathcal{E} in (2.22), one infers,

$$\text{for all } \lambda \in \mathcal{E} \text{ and all } \varepsilon > 0, \quad |(\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathcal{E}| > 0 \quad (2.90)$$

and hence,

$$\mathcal{E} \subseteq \overline{\mathcal{E}}^e. \quad (2.91)$$

Thus one obtains by (2.89) that

$$\sigma_{\text{ac}}(H) \supseteq \overline{\mathcal{E}}^e \supseteq \mathcal{E}. \quad (2.92)$$

Since $\sigma(H) = \mathcal{E}$ by (2.86), this proves (2.87).

Equations (2.10) and (2.26) imply

$$-1/g(\lambda + i0, x_0) = \pm 2i \operatorname{Im}[m_{\pm}(\lambda + i0, x_0)] \text{ for a.e. } \lambda \in \mathcal{E}. \quad (2.93)$$

Finally, combining (2.26), (2.93), and (B.76) then yields that $\sigma_{\text{ac}}(H) = \mathcal{E}$ has uniform spectral multiplicity two since

$$\text{for a.e. } \lambda \in \mathcal{E}, \quad 0 < \pm \operatorname{Im}[m_{\pm}(\lambda + i0, x_0)] < \infty. \quad (2.94)$$

□

That $\sigma(H)$ has uniform multiplicity equal to two is in accordance with Theorem 9.1 in Deift and Simon [17].

Although it was not necessary to use the following information in the proof of Theorem 2.9, we note that by (A.14), one has

$$\overline{\mathcal{E}}^e \subseteq \overline{\mathcal{E}} = \mathcal{E}. \quad (2.95)$$

Thus, combining (2.91) and (2.95) leads to

$$\mathcal{E} = \overline{\mathcal{E}}^e = \overline{\mathcal{E}}. \quad (2.96)$$

One can also give an alternative proof of (2.87) based on the reflectionless property of V as follows (still under the assumptions of Theorem 2.9):

Alternative proof of (2.87). Fix $x_0 \in \mathbb{R}$. Since $V \in \text{SY}(\mathcal{E})$, (2.27) yields that

$$\xi(\lambda, x_0) = \frac{1}{\pi} \operatorname{Im}[\ln(g(\lambda + i0, x_0))] = \frac{1}{2} \text{ for a.e. } \lambda \in \mathcal{E} \quad (2.97)$$

and hence

$$\operatorname{Re}[g(\lambda + i0, x_0)] = 0 \text{ for a.e. } \lambda \in \mathcal{E}. \quad (2.98)$$

If there exists a measurable set $A \subseteq \mathcal{E}$ of positive Lebesgue measure, $|A| > 0$, on which $\operatorname{Im}(g)$ vanishes, that is,

$$\operatorname{Im}[g(\lambda + i0, x_0)] = 0 \text{ for a.e. } \lambda \in A, \quad (2.99)$$

then (2.98) and (2.99) yield the existence of a subset $B \subseteq \mathcal{E}$ of positive Lebesgue measure, $|B| > 0$, such that

$$g(\lambda + i0, x_0) = 0 \text{ for a.e. } \lambda \in B. \quad (2.100)$$

Since $g(\cdot, x_0)$ is a Herglotz function, the uniqueness property of Herglotz functions in Theorem B.2(ii) yields the contradiction $g \equiv 0$. Thus, no such set $B \subseteq \mathcal{E}$ exists and one concludes that

$$\operatorname{Im}[g(\lambda + i0, x_0)] > 0 \text{ for a.e. } \lambda \in \mathcal{E}. \quad (2.101)$$

Moreover, since $g(\lambda + i0, x_0)$ exists finitely for a.e. $\lambda \in \mathbb{R}$ by Theorem B.2(i), this yields

$$0 < \operatorname{Im}[g(\lambda + i0, x_0)] < \infty \text{ for a.e. } \lambda \in \mathcal{E}. \quad (2.102)$$

By the asymptotic behavior of g in (2.19), this shows that

$$g(z, x_0) = [m_{-}(z, x_0) - m_{+}(z, x_0)]^{-1} \quad (2.103)$$

is a Herglotz function of the type (cf. (B.3), (B.9), and (B.60)–(B.62))

$$g(z, x_0) = \int_{\mathbb{R}} \frac{d\Omega_{0,0}(\lambda, x_0)}{\lambda - z}, \quad z \in \mathbb{C}_+. \quad (2.104)$$

Combining (2.102) and (B.5), the absolutely continuous part $d\Omega_{0,0,\text{ac}}(\cdot, x_0)$ of the measure $d\Omega_{0,0}(\cdot, x_0)$ is supported on \mathcal{E} ,

$$\text{supp}[d\Omega_{0,0,\text{ac}}(\cdot, x_0)] = \mathcal{E}. \quad (2.105)$$

(This also follows from combining, (2.101), (A.19), and (B.49).) Next, we replace m_{\pm} in (B.64), (B.65) by $m_{\pm}(\cdot, x_0)$ (cf. the paragraph preceding Theorem B.10) and introduce the trace measure $d\Omega^{\text{tr}}(\cdot, x_0)$ associated with the trace $M^{\text{tr}}(\cdot, x_0)$ of the 2×2 Weyl–Titchmarsh matrix $M(\cdot, x_0)$ of the Schrödinger operator H

$$M^{\text{tr}}(z, x_0) = M_{0,0}(z, x_0) + M_{1,1}(z, x_0) = \frac{1 + m_-(z, x_0)m_+(z, x_0)}{m_-(z, x_0) - m_+(z, x_0)} \quad (2.106)$$

$$= a(x_0) + \int_{\mathbb{R}} d\Omega^{\text{tr}}(\lambda, x_0) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \quad (2.107)$$

$$a(x_0) = \text{Re}[M^{\text{tr}}(i, x_0)], \quad d\Omega^{\text{tr}}(\cdot, x_0) = d\Omega_{0,0}(\cdot, x_0) + d\Omega_{1,1}(\cdot, x_0).$$

By (2.105) one infers that

$$\text{supp}[d\Omega_{\text{ac}}^{\text{tr}}(\cdot, x_0)] = \mathcal{E} \quad (2.108)$$

and hence

$$\sigma_{\text{ac}}(H) = \mathcal{E} \quad (2.109)$$

holds as a consequence of (B.66). \square

In addition to

$$M_{0,0}(z, x) = g(z, x) = [m_-(z, x) - m_+(z, x)]^{-1}, \quad z \in \mathbb{C}_+, \quad x \in \mathbb{R} \quad (2.110)$$

we will also analyze

$$M_{1,1}(z, x) = h(z, x) = \frac{m_-(z, x)m_+(z, x)}{m_-(z, x) - m_+(z, x)}, \quad z \in \mathbb{C}_+, \quad x \in \mathbb{R}. \quad (2.111)$$

Recalling $m_{\pm} = \psi'_{\pm}/\psi_{\pm}$ (cf. (2.49)), we note that h is given by

$$h(z, x) = \frac{\psi'_+(z, x, x)\psi'_-(z, x, x)}{W(\psi_+(z, x, x), \psi_-(z, x, x))}, \quad z \in \mathbb{C}_+, \quad x \in \mathbb{R}. \quad (2.112)$$

In analogy to the product expansion for g in (2.51) one then obtains

$$h(z, x) = \frac{i}{2} \frac{[z - \nu_0(x)]}{(z - E_0)^{1/2}} \prod_{j \in J} \frac{[z - \nu_j(x)]}{[(z - a_j)(z - b_j)]^{1/2}}, \quad z \in \mathbb{C} \setminus \mathcal{E}, \quad x \in \mathbb{R}, \quad (2.113)$$

where (in analogy to $\mu_j(x) \in [a_j, b_j]$, $j \in J$, $x \in \mathbb{R}$, in connection with $g(\cdot, x)$)

$$\nu_0(x) \leq E_0, \quad \nu_j(x) \in [a_j, b_j], \quad j \in J, \quad x \in \mathbb{R}. \quad (2.114)$$

Moreover, (2.18) and (2.111) imply the universal asymptotic expansion, valid for all $x \in \mathbb{R}$,

$$h(z, x) \underset{\substack{|z| \rightarrow \infty \\ z \in C_{\varepsilon}}}{=} (i/2)z^{1/2}[1 + o(1)]. \quad (2.115)$$

3. ABSENCE OF SINGULAR SPECTRUM

In this section we will prove the absence of the singular spectrum of a reflectionless Schrödinger operator H with potential V in the Sodin–Yuditskii class $\text{SY}(\mathcal{E})$, that is, we intend to prove that

$$\sigma(H) = \sigma_{\text{ac}}(H) = \mathcal{E}, \quad \sigma_{\text{sc}}(H) = \sigma_{\text{pp}}(H) = \emptyset. \quad (3.1)$$

Unlike our two elementary proofs of $\sigma_{\text{ac}}(H) = \mathcal{E}$ in Section 1 (cf. (2.87)), the proof of (3.1) relies on certain techniques developed in harmonic analysis and potential theory associated with domains $(\mathbb{C} \cup \{\infty\}) \setminus \mathcal{E}$.

We start with an elementary lemma which will permit us to reduce the discussion of unbounded homogeneous sets \mathcal{E} (typical for Schrödinger operators) to the case of a compact homogeneous sets $\tilde{\mathcal{E}}$ (as discussed by Peherstorfer and Yuditskii [61] and Sodin and Yuditskii [75] in connection with Jacobi operators).

Lemma 3.1. *Let m be a Herglotz function with representation*

$$\begin{aligned} m(z) &= c + \int_{\mathbb{R}} d\omega(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \\ c &= \text{Re}[m(i)], \quad \int_{\mathbb{R}} \frac{d\omega(\lambda)}{1 + \lambda^2} < \infty, \quad \mathbb{R} \setminus \text{supp}(d\omega) \neq \emptyset. \end{aligned} \quad (3.2)$$

Consider the change of variables

$$\begin{aligned} z &\mapsto \zeta = (\lambda_0 - z)^{-1}, \quad z = \lambda_0 - \zeta^{-1}, \quad z \in \mathbb{C} \cup \{\infty\}, \\ &\text{for some fixed } \lambda_0 \in \mathbb{R} \setminus \text{supp}(d\omega). \end{aligned} \quad (3.3)$$

Then,

$$\tilde{m}(\zeta) = m(z(\zeta)) = \tilde{c} + \int_{\mathbb{R}} \frac{d\tilde{\omega}(\eta)}{\eta - \zeta}, \quad \zeta \in \mathbb{C}_+, \quad (3.4)$$

$$d\tilde{\omega}(\eta) = x^2 d\omega(\lambda_0 - \eta^{-1})|_{\text{supp}(d\tilde{\omega})}, \quad (3.5)$$

$$\text{supp}(d\tilde{\omega}) \subseteq [-\text{dist}(\lambda_0, \text{supp}(d\omega))^{-1}, \text{dist}(\lambda_0, \text{supp}(d\omega))^{-1}], \quad (3.6)$$

$$\tilde{c} = c + \int_{\text{supp}(d\tilde{\omega})} d\tilde{\omega}(\eta) \frac{\lambda_0 - (1 + \lambda_0^2)\eta}{1 - 2\lambda_0\eta + (1 + \lambda_0^2)\eta^2}. \quad (3.7)$$

In particular, $d\tilde{\omega}$ is purely absolutely continuous if and only if $d\omega$ is, and

$$\text{if } d\omega(\lambda) = \omega'(\lambda)d\lambda|_{\text{supp}(d\omega)}, \text{ then } d\tilde{\omega}(\eta) = \omega'(\lambda_0 - \eta^{-1})d\eta|_{\text{supp}(d\tilde{\omega})}. \quad (3.8)$$

Proof. This is a straightforward computation. We note that

$$\int_{\mathbb{R}} \frac{d\omega(\lambda)}{1 + \lambda^2} < \infty \text{ is equivalent to } \int_{\mathbb{R}} d\tilde{\omega}(\eta) < \infty. \quad (3.9)$$

□

Next, we need the notion of a compact homogeneous set and hence slightly modify Definition 2.2 as follows.

Definition 3.2. Let $\tilde{\mathcal{E}} \subset \mathbb{R}$ be a compact set which we may write as

$$\tilde{\mathcal{E}} = [E_0, E_1] \setminus \bigcup_{j \in \tilde{J}} (\tilde{a}_j, \tilde{b}_j), \quad \tilde{J} \subseteq \mathbb{N}, \quad (3.10)$$

for some $E_0, E_1 \in \mathbb{R}$, $E_0 < E_1$, and $\tilde{a}_j < \tilde{b}_j$, where $(\tilde{a}_j, \tilde{b}_j) \cap (\tilde{a}_{j'}, \tilde{b}_{j'}) = \emptyset$, $j, j' \in \tilde{J}$, $j \neq j'$. Then $\tilde{\mathcal{E}}$ is called *homogeneous* if

$$\begin{aligned} &\text{there exists an } \varepsilon > 0 \text{ such that for all } \eta \in \tilde{\mathcal{E}} \\ &\text{and all } 0 < \delta < \text{diam}(\tilde{\mathcal{E}}), \quad |\tilde{\mathcal{E}} \cap (\eta - \delta, \eta + \delta)| \geq \varepsilon \delta. \end{aligned} \quad (3.11)$$

The following theorem is a direct consequence of a result of Zinsmeister [81, Theorem 1] (cf. Theorem B.7) and also a special case of a result of Peherstorfer and Yuditskii [61] proved by entirely different methods based on character automorphic Hardy functions.

Theorem 3.3 ([81], Theorem 1, [61], Lemma 2.4). *Let $\tilde{\mathcal{E}} \subset \mathbb{R}$ be a compact homogeneous set and r a Herglotz function with representation*

$$\begin{aligned} r(\zeta) &= a + \int_{\tilde{\mathcal{E}}} \frac{d\mu(\eta)}{\eta - \zeta}, \quad \zeta \in \mathbb{C}_+, \\ a &\in \mathbb{R}, \quad d\mu \text{ a finite measure, } \text{supp}(d\mu) \subseteq \tilde{\mathcal{E}}. \end{aligned} \quad (3.12)$$

Denote by $r(\eta + i0) = \lim_{\varepsilon \downarrow 0} r(\eta + i\varepsilon)$ the a.e. normal boundary values of r and assume that

$$\text{Re}[r(\cdot + i0)] \in L^1(\tilde{\mathcal{E}}; d\eta). \quad (3.13)$$

Then, $d\mu$ is purely absolutely continuous and hence

$$d\mu(\eta) = d\mu_{\text{ac}}(\eta) = \frac{1}{\pi} \text{Im}[r(\eta + i0)] d\eta. \quad (3.14)$$

Proof. We note that r is real-valued on $\mathbb{R} \setminus \tilde{\mathcal{E}}$ and $\text{Im}[r(\cdot + i0)] \in L^1_{\text{loc}}(\mathbb{R}; d\eta)$ by general principles. Since $\tilde{\mathcal{E}}$ is compact, one has $\text{Im}[r(\cdot + i0)] \in L^1(\mathbb{R}; d\eta)$. Thus, the integrability condition (3.13) is equivalent to

$$r(\cdot + i0) \in L^1(\tilde{\mathcal{E}}; d\eta). \quad (3.15)$$

Temporarily denoting by $d\eta$ the Lebesgue measure on \mathbb{R} , and introducing the auxiliary Herglotz function

$$w(\zeta) = a + \left[\int_{\tilde{\mathcal{E}}} d\mu(\eta) \right] / \left[\int_{\tilde{\mathcal{E}}} d\eta \right] \int_{\tilde{\mathcal{E}}} \frac{d\eta}{\eta - \zeta}, \quad \zeta \in \mathbb{C}_+, \quad (3.16)$$

we consider the function

$$F(\zeta) = r(\zeta) - w(\zeta) = \int_{\tilde{\mathcal{E}}} \frac{d\nu(\eta)}{\eta - \zeta}, \quad \zeta \in \mathbb{C}_+ \quad (3.17)$$

with the finite signed measure $d\nu$ given by

$$d\nu(\eta) = d\mu(\eta) - \left[\int_{\tilde{\mathcal{E}}} d\mu(\eta) \right] / \left[\int_{\tilde{\mathcal{E}}} d\eta \right] \chi_{\tilde{\mathcal{E}}}(\eta) d\eta. \quad (3.18)$$

Then F is analytic on $\mathbb{C} \setminus \tilde{\mathcal{E}}$ and $\lim_{|z| \rightarrow \infty} zF(z) = 0$. The symmetry property $F(\bar{z}) = \overline{F(z)}$, the integrability condition (3.15), and the known existence of the nontangential limits $r_{\pm}(\eta) = \lim_{\zeta \in \mathbb{C}_{\pm}, \zeta \rightarrow \eta} r(\zeta)$ for a.e. $\eta \in \tilde{\mathcal{E}}$, imply that

$$d\nu \in \widetilde{H}^1(\tilde{\mathcal{E}}) \quad (3.19)$$

(cf. the definition of $\widetilde{H}^1(\tilde{\mathcal{E}})$ in (B.44)). Theorem B.7 then proves that $d\nu$, and hence $d\mu$, is purely absolutely continuous. \square

Combining Lemma 3.1 and Theorem 3.3, we obtain the principal result of this section. (Incidentally, this yields yet another proof of (2.87).)

Theorem 3.4. *Let $V \in \text{SY}(\mathcal{E})$ and denote by H the associated Schrödinger operator defined in (2.85). Then, the spectrum of H is purely absolutely continuous and coincides with \mathcal{E} ,*

$$\sigma(H) = \sigma_{\text{ac}}(H) = \mathcal{E}, \quad \sigma_{\text{sc}}(H) = \sigma_{\text{pp}}(H) = \emptyset. \quad (3.20)$$

Moreover, $\sigma(H)$ has uniform multiplicity equal to two.

Proof. By Theorem 2.9 we need to prove the absence of the singular spectrum of H ,

$$\sigma_{\text{sc}}(H) = \sigma_{\text{pp}}(H) = \emptyset. \quad (3.21)$$

Recalling the absolutely convergent product expansion for the diagonal Green's function g ,

$$g(z, x) = \frac{i}{2(z - E_0)^{1/2}} \prod_{j \in J} \frac{[z - \mu_j(x)]}{[(z - a_j)(z - b_j)]^{1/2}}, \quad z \in \mathbb{C} \setminus \mathcal{E}, \quad x \in \mathbb{R}, \quad (3.22)$$

we pick $\lambda_0 < \nu_0(x)$ (implying $\lambda_0 < E_0$) and introduce the change of variables

$$z \mapsto \zeta = (\lambda_0 - z)^{-1}, \quad z = \lambda_0 - \zeta^{-1}, \quad z \in \mathbb{C} \cup \{\infty\}. \quad (3.23)$$

This results in

$$\tilde{g}(\zeta, x) = g(z(\zeta), x) = C(x) \frac{\zeta^{1/2}}{(\zeta - \tilde{E}_0)^{1/2}} \prod_{j \in J} \frac{[\zeta - \tilde{\mu}_j(x)]}{[(\zeta - \tilde{a}_j)(\zeta - \tilde{b}_j)]^{1/2}}, \quad \zeta \in \mathbb{C} \setminus \tilde{\mathcal{E}}, \quad x \in \mathbb{R}, \quad (3.24)$$

where

$$\begin{aligned} \tilde{E}_0 &= (\lambda_0 - E_0)^{-1}, \\ \tilde{a}_j &= (\lambda_0 - a_j)^{-1}, \quad \tilde{b}_j = (\lambda_0 - b_j)^{-1}, \quad j \in J, \\ \tilde{\mu}_j(x) &= [\lambda_0 - \mu_j(x)]^{-1}, \quad j \in J, \quad x \in \mathbb{R}, \end{aligned} \quad (3.25)$$

and

$$\tilde{\mathcal{E}} = [\tilde{E}_0, 0] \setminus \bigcup_{j \in J} (\tilde{a}_j, \tilde{b}_j). \quad (3.26)$$

Since

$$g(z, x) \underset{z \downarrow -\infty}{=} \frac{1}{2|z|^{1/2}} [1 + o(1)], \quad x \in \mathbb{R}, \quad (3.27)$$

one concludes

$$\tilde{g}(\zeta, x) \underset{\zeta \downarrow 0}{=} \frac{|\zeta|^{1/2}}{2} [1 + o(1)], \quad x \in \mathbb{R}, \quad (3.28)$$

and hence

$$C(x) > 0 \quad (3.29)$$

in (3.24). Next, we denote by $g(\lambda + i0, x) = \lim_{\varepsilon \downarrow 0} g(\lambda + i\varepsilon, x)$ and $\tilde{g}(\eta + i0, x) = \lim_{\varepsilon \downarrow 0} \tilde{g}(\eta + i\varepsilon, x)$ the a.e. normal boundary values of $g(\cdot, x)$ and $\tilde{g}(\cdot, x)$, $x \in \mathbb{R}$. By (2.39),

$$\text{for each } x \in \mathbb{R}, \quad g(\lambda + i0, x) \text{ is purely imaginary for a.e. } \lambda \in \mathcal{E} \quad (3.30)$$

and hence

$$\text{for each } x \in \mathbb{R}, \quad \tilde{g}(\eta + i0, x) \text{ is purely imaginary for a.e. } \eta \in \tilde{\mathcal{E}}. \quad (3.31)$$

In particular,

$$\text{for each } x \in \mathbb{R}, \quad \operatorname{Re}[\tilde{g}(\eta + i0, x)] = 0 \text{ for a.e. } \eta \in \tilde{\mathcal{E}} \quad (3.32)$$

and thus the analog of (3.13) is satisfied in the context of $\tilde{g}(\cdot, x)$. Applying Theorem 3.3 thus proves

$$\tilde{g}(\zeta, x) = C(x) + \frac{1}{\pi} \int_{\tilde{\mathcal{E}}} \frac{[-i\tilde{g}(\eta + i0, x)]d\eta}{\eta - \zeta}, \quad \zeta \in \mathbb{C} \setminus \tilde{\mathcal{E}}, \quad x \in \mathbb{R}. \quad (3.33)$$

Applying Lemma 3.1 to $g(\cdot, x)$ as represented in (2.104),

$$g(z, x) = \int_{\mathcal{E}} \frac{d\Omega_{0,0}(\lambda, x)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathcal{E}, \quad x \in \mathbb{R}, \quad (3.34)$$

then yields for the measure $d\Omega_{0,0}(\cdot, x)$,

$$\begin{aligned} \text{for each } x \in \mathbb{R}, \quad d\Omega_{0,0}(\lambda, x) &= d\Omega_{0,0,\text{ac}}(\lambda, x) = \frac{1}{\pi} [-ig(\lambda + i0, x)]d\lambda, \\ \operatorname{supp}(d\Omega_{0,0}(\cdot, x)) &= \mathcal{E}. \end{aligned} \quad (3.35)$$

Next, we repeat this analysis for $h(\cdot, x)$. Introducing,

$$\tilde{\nu}_0(x) = [\lambda_0 - \nu_0(x)]^{-1}, \quad \tilde{\nu}_j(x) = [\lambda - \nu_j(x)]^{-1}, \quad j \in J, \quad x \in \mathbb{R}, \quad (3.36)$$

$h(\cdot, x)$ transforms into

$$\begin{aligned} \tilde{h}(\zeta, x) = h(z(\zeta), x) &= -D(x) \frac{[\zeta - \tilde{\nu}_0(x)]}{\zeta^{1/2}(\zeta - \tilde{E}_0)^{1/2}} \prod_{j \in J} \frac{[\zeta - \tilde{\nu}_j(x)]}{[(\zeta - \tilde{a}_j)(\zeta - \tilde{b}_j)]^{1/2}}, \\ &\zeta \in \mathbb{C} \setminus \tilde{\mathcal{E}}, \quad x \in \mathbb{R}. \end{aligned} \quad (3.37)$$

The asymptotic behavior (2.115) then proves $D(x) > 0$. At this point one can again apply Lemma 3.1 and Theorem 3.3 to $\tilde{h}(\cdot, x)$ and obtains

$$\begin{aligned} \text{for each } x \in \mathbb{R}, \quad d\Omega_{1,1}(\lambda, x) &= d\Omega_{1,1,\text{ac}}(\lambda, x) = \frac{1}{\pi} [-ih(\lambda + i0, x)]d\lambda, \\ \operatorname{supp}(d\Omega_{1,1}(\cdot, x)) &= \mathcal{E}. \end{aligned} \quad (3.38)$$

Combining (3.35) and (3.38) then yields

$$\begin{aligned} d\Omega^{\text{tr}}(\cdot, x_0) &= d\Omega_{\text{ac}}^{\text{tr}}(\cdot, x_0) = \frac{1}{\pi} [-iM^{\text{tr}}(\lambda + i0, x_0)]d\lambda, \\ \operatorname{supp}[d\Omega^{\text{tr}}(\cdot, x_0)] &= \mathcal{E} \end{aligned} \quad (3.39)$$

for the trace measure $d\Omega^{\text{tr}}(\cdot, x_0)$ associated with H . By (B.66) this completes the proof of (3.21). \square

4. EXAMPLES OF REFLECTIONLESS SCHRÖDINGER OPERATORS WITH A SINGULAR COMPONENT IN THEIR SPECTRUM

In our final section we will construct examples of reflectionless Schrödinger operators with a singular component in their spectra by weakening the hypothesis that $\mathcal{E} \subset \mathbb{R}$ is a homogeneous set. Instead, we will assume the spectrum to be the complement of a Denjoy–Widom-type domain in \mathbb{C} .

We start by taking a closer look at Herglotz functions closely related to reflectionless Schrödinger operators.

Let \mathfrak{E} be a closed subset of the nonnegative real axis which we write as

$$\mathfrak{E} = [0, \infty) \setminus \bigcup_{j \in \mathbb{N}} (a_j, b_j). \quad (4.1)$$

To a positive measure $d\sigma$ supported on \mathfrak{E} , such that

$$\int_{\mathfrak{E}} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty, \quad (4.2)$$

we associate the Herglotz function

$$r(z) = \operatorname{Re}(r(i)) + \int_{\mathfrak{E}} d\sigma(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \quad (4.3)$$

and recall the property $\operatorname{Im}(r(z)) > 0$, $z \in \mathbb{C}_+$. In this note we consider a specific subclass of such functions which is directly related to the diagonal terms of the 2×2 Weyl–Titchmarsh matrix of reflectionless one-dimensional Schrödinger operators and hence of particular importance to spectral theory. Indeed, motivated by the property (2.39) of the diagonal Green's function $g(\cdot, x)$, $x \in \mathbb{R}$, being purely imaginary a.e. on \mathcal{E} , we introduce the following definition.

Definition 4.1. Let r be a Herglotz function and \mathfrak{E} be a closed subset of $[0, \infty)$ of the type (4.1). Then r is said to belong to the class $\mathcal{R}(\mathfrak{E})$ if

$$r(\lambda + i0) \in i\mathbb{R} \text{ for a.e. } \lambda \in \mathfrak{E}. \quad (4.4)$$

We note that r is increasing on every interval (a_j, b_j) and thus there exist unique $\mu_j \in [a_j, b_j]$, $j \in \mathbb{N}$, such that

$$\begin{cases} r(\lambda) \leq 0, & a_j \leq \lambda \leq \mu_j, \\ r(\lambda) \geq 0, & \mu_j \leq \lambda \leq b_j, \end{cases} \quad j \in \mathbb{N}. \quad (4.5)$$

Together with (4.4) this implies that the argument $\xi(\lambda)$ of $r(\lambda + i0)$, that is,

$$\xi(\lambda) = \operatorname{Im}(\log(r(\lambda + i0))) \quad (4.6)$$

is defined for a.e. $\lambda \in \mathbb{R}$, and one obtains the exponential Herglotz representation of r ,

$$\log(r(z)) = \log(C) + \int_0^\infty d\lambda \xi(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad (4.7)$$

where

$$\xi(\lambda) = \begin{cases} 1, & a_j \leq \lambda \leq \mu_j, \\ 0, & \mu_j \leq \lambda \leq b_j, \\ 1/2, & \lambda \in \mathfrak{E}. \end{cases} \quad (4.8)$$

Next, we turn to the case where the measure $d\sigma$ in the Herglotz representation (4.3) of r also satisfies the condition

$$\int_{\mathfrak{E}} \frac{d\sigma(\lambda)}{1 + |\lambda|} < \infty, \quad (4.9)$$

and where r in (4.3) is of the special form

$$r(z) = \int_{\mathfrak{E}} \frac{d\sigma(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+. \quad (4.10)$$

Henceforth, we will assume that the sum of all lengths of the intervals (a_j, b_j) is finite, that is,

$$\sum_{j \in \mathbb{N}} (b_j - a_j) < \infty. \quad (4.11)$$

In this case the representation (4.7) can be rewritten in the form

$$r(z) = \frac{i}{2z^{1/2}} \prod_{j \in \mathbb{N}} \frac{(z - \mu_j)}{[(z - a_j)(z - b_j)]^{1/2}}, \quad (4.12)$$

where the constant C in (4.7) is chosen such that

$$r(z) \underset{z \downarrow -\infty}{=} (i/2)z^{-1/2}[1 + o(1)]. \quad (4.13)$$

Under the assumption of a certain regularity of the set \mathfrak{E} , there exists a specific choice of the zeros $\{\mu_j\}_{j \in \mathbb{N}}$ in (4.12) which will lead to a purely absolutely continuous measure in the Herglotz representation (4.3). To identify this particular choice of zeros, we denote by

$$m(z) = \frac{i}{2z^{1/2}} \prod_{j \in \mathbb{N}} \frac{(z - c_j)}{[(z - a_j)(z - b_j)]^{1/2}}, \quad z \in \mathbb{C}_+, \quad (4.14)$$

a Herglotz function such that

$$-\int_{a_j+0}^{c_j} dz m(z) = \int_{c_j}^{b_j-0} dz m(z) < \infty, \quad j \in \mathbb{N}, \quad (4.15)$$

and, in addition,

$$\int_{-1}^{-0} dz m(z) < \infty. \quad (4.16)$$

Next, we make an even stronger assumption and suppose that for the above choice of $\{c_j\}_{j \in \mathbb{N}}$, the following *Parreau–Widom-type* condition (cf. [79], [80]) also holds,

$$\sum_{j \in \mathbb{N}} \int_{c_j}^{b_j-0} dz m(z) < \infty. \quad (4.17)$$

Remark 4.2. In terms of the conformal map (2.57), we have

$$\Theta(z) = \frac{1}{i} \int_0^z dz m(z), \quad z \in \mathbb{C}_+, \quad (4.18)$$

and $c_j = \Theta^{-1}(u_j + ih_j)$, $j \in \mathbb{N}$, that is, $\sum_{j \in \mathbb{N}} h_j < \infty$ becomes (4.17).

Theorem 4.3. *Suppose the Herglotz function m introduced in (4.14) satisfies conditions (4.15)–(4.17). Then the measure $d\rho$ in the Herglotz representation of m ,*

$$m(z) = \int_{\mathfrak{E}} \frac{d\rho(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+, \quad (4.19)$$

is absolutely continuous with respect to Lebesgue measure $d\lambda$ on \mathbb{R} .

Proof. We recall that any (nonvanishing) Herglotz function is a function of the *Smirnov class* $N_+(\mathbb{C}_+)$ (i.e., a ratio of two uniformly bounded functions such that the denominator is an outer function). In fact, any Herglotz function is an outer function (cf. [66, p. 111]). The following maximum principle holds for functions $g \in N_+(\mathbb{C}_+)$: $g(\cdot + i0) \in L^p(\mathbb{R}; d\lambda)$ implies $g \in H^p(\mathbb{C}_+)$. Next, we will show that $m(\cdot + i0)/(\cdot + i) \in L^1(\mathbb{R}; d\lambda)$.

Due to (4.16) and (4.17) the function $m(\cdot + i0)/(\cdot + i)$ is integrable on $(-1, 0)$ and on $\bigcup_{j \in \mathbb{N}}(a_j, b_j)$. Next, since $m(\lambda + i0) \in i\mathbb{R}$ for a.e. $\lambda \in \mathfrak{E}$ and thus, $m(\lambda + i0) = i\pi\rho'_{\text{a.c.}}(\lambda)$ a.e. on \mathfrak{E} , we get

$$\frac{1}{\pi i} \int_{\mathfrak{E}} \frac{m(\lambda + i0)d\lambda}{1 + |\lambda|} \leq \int_{\mathfrak{E}} \frac{d\rho(\lambda)}{1 + |\lambda|} < \infty. \quad (4.20)$$

Finally, by (4.13) the function $m(\cdot + i0)/(\cdot + i)$ is integrable on $(-\infty, -1]$. Thus,

$$\int_{\mathbb{R}} \frac{|m(\lambda + i0)|d\lambda}{1 + |\lambda|} < \infty. \quad (4.21)$$

Next, following an argument in [66, p. 96–97], we introduce the function

$$n(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{m(\lambda + i0)d\lambda}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (4.22)$$

The n is analytic on $\mathbb{C}_+ \cup \mathbb{C}_-$ and

$$\begin{aligned} n(z) - n(\bar{z}) &= \frac{1}{2\pi i} \int_{\mathbb{R}} m(\lambda + i0)d\lambda \left(\frac{1}{\lambda - z} - \frac{1}{\lambda - \bar{z}} \right) \\ &= \frac{y}{\pi} \int_{\mathbb{R}} \frac{m(\lambda + i0)d\lambda}{(\lambda - x)^2 + y^2} \\ &= m(z), \quad z = x + iy, \ y > 0, \end{aligned} \quad (4.23)$$

where we used Fatou's theorem in the last step (cf. [66, p. 86]). Since n and m are analytic on \mathbb{C}_+ , so is $n(\bar{z})$, $z \in \mathbb{C}_+$. But then,

$$\bar{n}(\bar{z}) = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\overline{m(\lambda + i0)}d\lambda}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (4.24)$$

is also analytic on \mathbb{C}_+ and hence $n(\bar{z})$ must be constant for $z \in \mathbb{C}_+$. Since

$$\lim_{y \uparrow \infty} n(-iy) = 0, \quad (4.25)$$

(using (4.21) and applying the dominated convergence theorem), one concludes

$$n(\bar{z}) = 0, \quad z \in \mathbb{C}_+, \quad (4.26)$$

and hence,

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{m(\lambda + i0)d\lambda}{\lambda - z} = m(z), \quad \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{m(\lambda + i0)d\lambda}{\lambda - \bar{z}} = 0, \quad z \in \mathbb{C}_+. \quad (4.27)$$

Thus, since m is real-valued on $\mathbb{R} \setminus \mathfrak{E}$,

$$\begin{aligned} m(z) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{[m(\lambda + i0) - \overline{m(\lambda + i0)}]d\lambda}{\lambda - z} \\ &= \frac{1}{\pi} \int_{\mathfrak{E}} \frac{\text{Im}(m(\lambda + i0))d\lambda}{\lambda - z} \\ &= \int_{\mathfrak{E}} \frac{d\rho(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+, \end{aligned} \quad (4.28)$$

and hence, $d\rho(\lambda) = \pi^{-1} \text{Im}(m(\lambda + i0))d\lambda$. \square

The goal of this section is to show that the above Parreau–Widom-type condition is insufficient in guaranteeing that all measures related to the class $\mathcal{R}(\mathfrak{E})$ are absolutely continuous. More precisely, we will show that Herglotz functions r of the type (4.12), for a certain distribution of its zeros $\{\mu_j\}_{j \in \mathbb{N}}$, may have a singular component.

To this end, we proceed to constructing a closed set \mathfrak{E} such that the intervals (a_j, b_j) , $j \in \mathbb{N}$, accumulate just at a single point, for instance, the origin,

$$0 < \cdots < a_{j+1} < b_{j+1} < a_j < b_j < \cdots < a_2 < b_2 < a_1 < b_1 < \infty. \quad (4.29)$$

In this case, the origin is the only point that may support a singular component, $\sigma(\{0\}) > 0$. A good strategy for accomplishing this goal is to put all zeros μ_j in (4.12) as far as possible from the origin, that is, we will choose $\mu_j = b_j$, $j \in \mathbb{N}$, in the next lemma.

Lemma 4.4. *Suppose \mathfrak{E} is of the form (4.1) with $\{a_j\}_{j \in \mathbb{N}}$ and $\{b_j\}_{j \in \mathbb{N}}$ satisfying (4.29). In addition, assume that the product $\prod_{j \in \mathbb{N}} (b_{j+1}/a_j)$ converges absolutely, that is, suppose*

$$\sum_{j \in \mathbb{N}} [1 - (b_{j+1}/a_j)] < \infty. \quad (4.30)$$

Define the Herglotz function r_0 by

$$r_0(z) = \frac{i}{2z^{1/2}} \prod_{j \in \mathbb{N}} \left(\frac{z - b_j}{z - a_j} \right)^{1/2} = \int_{\mathfrak{E}} \frac{d\sigma_0(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+. \quad (4.31)$$

Then the measure $d\sigma_0$ has a point mass at 0, that is,

$$\sigma_0(\{0\}) > 0. \quad (4.32)$$

Proof. First, we recall that

$$\sigma_0(\{0\}) = \lim_{z \uparrow 0} (-z)r_0(z) \quad (4.33)$$

since $d\sigma_0$ is supported on $\mathfrak{E} \subseteq [0, \infty)$. For a fixed $n \in \mathbb{N}$, we split the infinite product in (4.31) into three factors

$$\prod_{j=n+1}^{\infty} \left(\frac{z - b_j}{z - a_j} \right)^{1/2} \left(\frac{z - b_1}{z - a_n} \right)^{1/2} \prod_{j=1}^{n-1} \left(\frac{z - b_{j+1}}{z - a_j} \right)^{1/2}. \quad (4.34)$$

We note that for $z < 0$,

$$\frac{z - b_j}{z - a_j} > 1 \quad (4.35)$$

and

$$\frac{z - b_{j+1}}{z - a_j} > \frac{b_{j+1}}{a_j}. \quad (4.36)$$

Therefore,

$$-zr_0(z) > \frac{1}{2} \left[\frac{z(b_1 - z)}{z - a_n} \right]^{1/2} \prod_{j=1}^{n-1} \left(\frac{b_{j+1}}{a_j} \right)^{1/2} \geq \frac{1}{2} \left[\frac{z(b_1 - z)}{z - a_n} \right]^{1/2} \prod_{j \in \mathbb{N}} \left(\frac{b_{j+1}}{a_j} \right)^{1/2}. \quad (4.37)$$

Passing to the limit $n \rightarrow \infty$, we get

$$-zr_0(z) \geq \frac{1}{2} (b_1 - z)^{1/2} \prod_{j \in \mathbb{N}} (b_{j+1}/a_j)^{1/2}. \quad (4.38)$$

Then, passing to the limit $z \uparrow 0$, one obtains

$$\sigma_0(\{0\}) = \lim_{z \uparrow 0} (-z)r_0(z) \geq \frac{1}{2}b_1^{1/2} \prod_{j \in \mathbb{N}} (b_{j+1}/a_j)^{1/2} > 0. \quad (4.39)$$

□

Next, we turn to a construction of intervals symmetric with respect to the origin and to a given half-axis. Let $[b_1, \infty)$, $b_1 > 0$, be the given half-axis. We now find an interval $[b_2, a_1]$, which is symmetric with respect to the origin and to the half-axis $[b_1, \infty)$. Starting with intervals $[-1, -\alpha] \cup [\alpha, 1]$ under the conformal map $\zeta \mapsto z$, $\zeta = -k/z + 1$, $0 < k < b_1$, one gets

$$\alpha = -\frac{k}{b_1} + 1, \quad -1 = -\frac{k}{b_2} + 1, \quad -\alpha = -\frac{k}{a_1} + 1. \quad (4.40)$$

In particular

$$\frac{1}{b_2} - \frac{1}{a_1} = \frac{1}{b_1}. \quad (4.41)$$

Proceeding iteratively, one proves the following result.

Lemma 4.5. *Let $\mathfrak{E}_n = [b_1, \infty) \cup \bigcup_{j=2}^n [b_j, a_{j-1}] \subset [0, \infty)$, $n \geq 2$, be a finite system of closed intervals. Define an interval $[b_{n+1}, a_n]$ symmetric with respect to the origin and to the given half-axis $[b_n, \infty)$. Put $\mathfrak{E}_{n+1} = [b_1, \infty) \cup \bigcup_{j=2}^{n+1} [b_j, a_{j-1}]$. Let u_n be the solution of the Dirichlet problem $\Delta u = 0$ in $\mathbb{C} \setminus \mathfrak{E}_{n+1}$ with the boundary conditions $u|_{[b_{n+1}, a_n]} = 1$, $u|_{\mathfrak{E}_n} = 0$. Then,*

$$u_n(0) \geq 1/2. \quad (4.42)$$

Proof. Let u_0 correspond to the extremal case $\mathfrak{E} = [b_n, \infty)$. Due to the symmetry between $[b_{n+1}, a_n]$ and $[b_n, \infty)$, $u_0(0) = 1/2$. Note that the difference $u - u_0$ is nonnegative on the boundary of the domain $\mathbb{C} \setminus ([b_{n+1}, a_n] \cup [b_n, \infty))$ and therefore in its interior. Thus, $u(0) \geq u_0(0) = 1/2$. □

Next we turn to some properties of the corresponding function Θ in (4.18): Let Θ_n be the corresponding functions related to the system of intervals \mathfrak{E}_n introduced in Lemma 4.5. We recall that its imaginary part,

$$\omega_n(z) = \text{Im}(\Theta_n(z)), \quad z \in \mathbb{C} \setminus \mathfrak{E}_n, \quad (4.43)$$

is a positive single-valued harmonic function on the domain $\mathbb{C} \setminus \mathfrak{E}_n$ such that $\omega_n|_{\mathfrak{E}_n} = 0$ with the only singularity

$$\omega_n(z) \underset{z \rightarrow -\infty}{=} \text{Im}(z^{1/2})[1 + o(1)]. \quad (4.44)$$

Thus, $\omega_n(z) - \omega_{n+1}(z)$ is a uniformly bounded harmonic function (the singularity at infinity cancels). Since

$$\omega_n(\lambda \pm i0) - \omega_{n+1}(\lambda \pm i0) \geq 0, \quad \lambda \in \mathfrak{E}_{n+1}, \quad (4.45)$$

the same inequality also holds inside the domain

$$\omega_n(z) - \omega_{n+1}(z) \geq 0, \quad z \in \mathbb{C} \setminus \mathfrak{E}_{n+1}. \quad (4.46)$$

We also note that $\omega_n(\lambda)$ decreases in $(-\infty, b_n]$. In fact, $\Theta_n(\lambda) = i\omega_n(\lambda)$ for all $\lambda \in \mathbb{R} \setminus \mathfrak{E}_n$.

Theorem 4.6 (Construction of a Denjoy–Widom-type domain $\mathbb{C} \setminus \mathfrak{E}$).

There exists a closed set \mathfrak{E} of the form (4.1) with $\{a_j\}_{j \in \mathbb{N}}$ and $\{b_j\}_{j \in \mathbb{N}}$ satisfying (4.29), such that for the choice $\mu_j = b_j$, $j \in \mathbb{N}$, the corresponding Herglotz function r_0 defined in (4.31) leads to a Herglotz representation, where the associated measure $d\sigma_0$ has a point mass at 0, whereas the choice $\mu_j = c_j$, $j \in \mathbb{N}$, leads to a Herglotz function m defined in (4.14) with corresponding measure $d\rho$ in (4.19) purely absolutely continuous with respect to Lebesgue measure.

Proof. We follow closely the construction given by Hasumi [35, p. 215–218]. Starting with $[b_1, \infty)$, $b_1 > 0$, we will construct a system of intervals accumulating at the origin such that on one hand the series (4.30) converges and such that, on the other hand, the measure $d\rho$ (see Theorem 4.3) is absolutely continuous.

First we claim that the following three conditions can be simultaneously satisfied by choosing ℓ_n , $0 < \ell_n < 1/2$, $n \in \mathbb{N}$, sufficiently small:

$$\begin{aligned} (a) \quad & b_{n+1} = \ell_n b_n, \\ (b) \quad & \frac{1}{b_{n+1}} - \frac{1}{a_n} = \frac{1}{b_n}, \\ (c) \quad & \inf_{n \in \mathbb{N}} \{\omega_n(z) \mid z \in [b_{n+1}, a_n]\} \geq \frac{1}{2} \omega_n(0). \end{aligned} \tag{4.47}$$

Indeed, making ℓ_n smaller, we get an interval $[b_{n+1}, a_n]$ that approaches the origin. Thus, conditions (a) and (b) hold. In addition, by continuity of ω_n , condition (c) is also satisfied.

We note that under this construction, condition (4.30) is of the form

$$\sum_{j \in \mathbb{N}} b_{j+1} \left(\frac{1}{b_{j+1}} - \frac{1}{a_j} \right) = \sum_{j \in \mathbb{N}} b_{j+1} \frac{1}{b_j} = \sum_{j \in \mathbb{N}} \ell_j. \tag{4.48}$$

Thus, if necessary, by making ℓ_n even smaller in a such way that $\sum_{n \in \mathbb{N}} \ell_n < \infty$, condition (4.30) is satisfied.

As a next step we seek an inductive estimate on $\omega_n(0)$. Due to condition (c) in (4.47) one infers

$$\omega_n(z) - \omega_{n+1}(z) - \frac{1}{2} \omega_n(0) u_n(z) \geq 0, \quad z \in \mathfrak{E}_{n+1}, \tag{4.49}$$

where u_n is the solution of the Dirichlet problem $\Delta u = 0$ in $\mathbb{C} \setminus \mathfrak{E}_{n+1}$ with the boundary conditions $u|_{[b_{n+1}, a_n]} = 1$, $u|_{\mathfrak{E}_n} = 0$. Since the same inequality holds inside the domain, using (4.42) one gets,

$$\omega_n(0) - \omega_{n+1}(0) \geq \frac{1}{2} \omega_n(0) u_n(0) \geq \frac{1}{4} \omega_n(0). \tag{4.50}$$

Thus, $\{\omega_n(0)\}_{n \in \mathbb{N}}$ form a sequence that is dominated by a geometric progression,

$$\omega_{n+1}(0) \leq \frac{3}{4} \omega_n(0) \leq \left(\frac{3}{4} \right)^{n+1} \omega_0(0), \quad n \in \mathbb{N}_0. \tag{4.51}$$

Now we are in position to show that all assumptions of Theorem 4.3 are satisfied. We note that

$$\omega(z) = \lim_{n \uparrow \infty} \omega_n(z), \tag{4.52}$$

and thus

$$m(z) = \lim_{n \uparrow \infty} m_n(z), \quad \Theta(z) = \lim_{n \uparrow \infty} \Theta_n(z). \tag{4.53}$$

Due to the monotonicity property with respect to n one gets

$$\omega(z) \leq \omega_n(z), \quad (4.54)$$

in particular,

$$\omega(c_n) \leq \omega_n(c_n), \quad (4.55)$$

and due to the monotonicity property of $\omega_n(\cdot)$ on $(-\infty, b_n]$ one obtains (cf. (4.51))

$$\omega_n(c_n) \leq \omega_n(0) \leq \left(\frac{3}{4}\right)^n \omega_0(0), \quad n \in \mathbb{N}. \quad (4.56)$$

Thus,

$$\sum_{n \in \mathbb{N}} \omega(c_n) < \infty, \quad (4.57)$$

that is, (4.17) is satisfied. Finally,

$$\begin{aligned} \int_{-1}^{-0} dz m(z) &= \int_{-1}^{-0} dz \lim_{n \uparrow \infty} m_n(z) \leq \lim_{n \uparrow \infty} \int_{-1}^{-0} dz m_n(z) \\ &= \lim_{n \uparrow \infty} [\omega_n(-1) - \omega_n(0)] \leq \omega(-1) < \infty. \end{aligned} \quad (4.58)$$

□

Remark 4.7. By inspection (cf. Theorem B.8), the measure $d\sigma_0$ in the Herglotz representation of r_0 in Theorem 4.6 has purely absolutely continuous spectrum away from zero. This is in agreement with a result of Aronszajn and Donoghue [3] recorded in Theorem B.4 (iv) since $\xi(\lambda) = 1/2$ on $\bigcup_{j \in \mathbb{N}} (b_{j+1}, a_j) \cup (b_1, \infty)$. This result by Aronszajn and Donoghue applies to open intervals and hence does not exclude an eigenvalue at $\lambda = 0$ (i.e., $\sigma_0(\{0\}) > 0$) as constructed in Theorem 4.6.

In order to apply this construction to one-dimensional Schrödinger operators we next recall sufficient conditions for $m_+(z) = m_+(z, 0)$, $z \in \mathbb{C}_+$, to be the half-line Weyl–Titchmarsh function associated with a Schrödinger operator on $[0, \infty)$ in terms of the corresponding measure $d\omega_+$ in the Herglotz representation of m_+ . Based on the classical inverse spectral theory approach due to Gelfand and Levitan [24], the following result discussed in Thurlow [78] (see also, [52, Sects. 2.5, 2.9], [53], [57, Sect. 26.5], [65]) describes sufficient conditions for a monotonically non-decreasing function ω_+ on \mathbb{R} to be the spectral function of a half-line Schrödinger operator H_+ in $L^2([0, \infty); dx)$ with a Dirichlet boundary condition at $x = 0$.

Theorem 4.8 ([78]). *Let ω_+ be a monotonically nondecreasing function on \mathbb{R} satisfying the following two conditions.*

(i) *Whenever $f \in C([0, \infty))$ with compact support contained in $[0, \infty)$ and*

$$\int_{\mathbb{R}} d\omega_+(\lambda) |F(\lambda)|^2 = 0, \quad \text{then } f = 0, \quad (4.59)$$

where

$$F(\lambda) = \int_0^\infty dx \frac{\sin(\lambda^{1/2}(x - x_0))}{\lambda^{1/2}} f(x), \quad \lambda \in \mathbb{R}. \quad (4.60)$$

(ii) *Define*

$$\tilde{\omega}_+(\lambda) = \begin{cases} \omega_+(\lambda) - \frac{2}{3\pi} \lambda^{3/2}, & \lambda \geq 0, \\ \omega_+(\lambda), & \lambda < 0 \end{cases} \quad (4.61)$$

and assume the limit

$$\lim_{R \uparrow \infty} \int_{-\infty}^R d\tilde{\omega}_+(\lambda) \frac{\sin(\lambda^{1/2}x)}{\lambda^{1/2}} = \Phi(x), \quad x \geq 0, \quad (4.62)$$

exists with $\Phi \in L^\infty([0, R]; dx)$ for all $R > 0$. Moreover, suppose that for some $r \in \mathbb{N}_0$, $\Phi^{(r+1)} \in L^1([0, R]; dx)$ for all $R > 0$, and $\lim_{x \downarrow 0} \Phi(x) = 0$.

Then $d\omega_+$ is the spectral measure of a self-adjoint Schrödinger operator H_+ in $L^2([0, \infty); dx)$ associated with the differential expression $L_+ = -d^2/dx^2 + V_+$, $x > 0$, with a Dirichlet boundary condition at $x = 0$, a self-adjoint boundary condition at ∞ (if necessary), and a real-valued potential coefficient V_+ satisfying $V_+^{(r)} \in L^1([0, R]; dx)$ for all $R > 0$.

Since the analogous result applies to Schrödinger operators on the half-line $(-\infty, 0]$, we omit the details.

Remark 4.9. We add two observations to Theorem 4.8: First, whenever the points of increase of ω_+ have a finite limit point, in particular, if ω_+ is strictly increasing on an interval, then condition (i) in Theorem 4.8 is satisfied. This follows from the fact that F is an entire function of order 1/2 which cannot vanish on a set with a finite accumulation point, or even on an interval, without vanishing identically. The latter yields $f = 0$ by the inverse sine transform. Secondly, if the minimal operator $H_{+, \min}$ associated with L_+ and a Dirichlet boundary condition at $x = 0$ is bounded from below, then L_+ is in the limit point case at infinity (i.e., $H_{+, \min}$ is essentially self-adjoint) and no boundary condition is needed at $x = \infty$ (this is again based on the well-known result of Hartman [34] (see also Rellich [64] and [25]). Here $H_{+, \min}$ in $L^2([0, \infty); dx)$ is defined as

$$\begin{aligned} H_{+, \min} f &= L_+ f, \\ f \in \text{dom}(H_{+, \min}) &= \{g \in L^2([0, \infty); dx) \mid \text{supp}(g) \text{ compact}; \\ &\quad \lim_{\varepsilon \downarrow 0} g(\varepsilon) = 0; L_+ g \in L^2([0, \infty); dx)\}. \end{aligned} \quad (4.63)$$

In the concrete situations we have in mind below, both observations clearly apply (as all spectra involved are bounded from below and contain intervals of absolutely continuous spectrum) and hence we will focus exclusively on condition (ii) in Theorem 4.8 and utilize the fact that all (half-line and full-line) Schrödinger operators considered in this paper are in the limit point case at $\pm\infty$.

In connection with Theorem 4.8 (ii), the following special case of Lemma 8.3.2 in [52] turns out to be useful.

Lemma 4.10 ([52], Lemma 8.3.2). *Let $f \in L^1([1, \infty); dp)$ and suppose for some $R > 1$ and $p > R$, f permits the asymptotic expansion*

$$f(p) \underset{p \uparrow \infty}{=} \sum_{k=1}^N f_k p^{-2k} + O(p^{-(2N+2)}) \quad \text{for all } N \in \mathbb{N} \quad (4.64)$$

for some coefficients $\{f_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$. Then the function

$$\phi(x) = \int_1^\infty dp \frac{\sin(px)}{p} f(p), \quad x \geq 0, \quad (4.65)$$

satisfies

$$\phi \in C^\infty((0, \infty)), \quad \phi^{(m)} \in L^\infty([0, \infty); dx) \quad \text{for all } m \in \mathbb{N}_0, \quad \lim_{x \downarrow 0} \phi(x) = 0, \quad (4.66)$$

and

$$\lim_{x \uparrow \infty} \phi^{(m)}(x) = 0 \text{ for all } m \in \mathbb{N}_0. \quad (4.67)$$

Next, we construct a one-dimensional Schrödinger operator H_{r_0} with spectrum the set \mathfrak{E} in Theorem 4.6, whose diagonal Green's function $g_{r_0}(\cdot, 0)$ coincides with the function r_0 in Theorem 4.6, and whose point spectrum is nonempty.

Theorem 4.11. *Let \mathfrak{E} be a closed set of the form (4.1) with $\{a_j\}_{j \in \mathbb{N}}$ and $\{b_j\}_{j \in \mathbb{N}}$ satisfying (4.29) and let r_0 be the Herglotz function defined in (4.31) with associated measure $d\sigma_0$ having a point mass at 0 as constructed in Theorem 4.6. Then, there exists a self-adjoint Schrödinger operator H_{r_0} in $L^2(\mathbb{R}; dx)$ with real-valued reflectionless potential coefficient V_{r_0} satisfying*

$$V_{r_0} \in C^\infty(\mathbb{R} \setminus \{0\}), \quad V_{r_0}^{(m)} \in L^1_{\text{loc}}(\mathbb{R}; dx) \text{ for all } m \in \mathbb{N}_0 \quad (4.68)$$

with spectral properties

$$\sigma(H_{r_0}) = \sigma_{\text{ac}}(H_{r_0}) = \mathfrak{E}, \quad \sigma_{\text{pp}}(H_{r_0}) = \{0\}, \quad \sigma_{\text{sc}}(H_{r_0}) = \emptyset. \quad (4.69)$$

In particular, H_{r_0} has nonempty singular component (a zero eigenvalue) in its spectrum.

Proof. First we note that if H is a reflectionless Schrödinger operator with associated diagonal Green's function $g(\cdot, x) \in \mathcal{R}(\mathfrak{E})$, $x \in \mathbb{R}$, then also

$$-g(\cdot, x)^{-1} \in \mathcal{R}(\mathfrak{E}), \quad x \in \mathbb{R}. \quad (4.70)$$

Moreover, by (2.10), we also have

$$-g(z, x)^{-1} = m_+(z, x) - m_-(z, x), \quad (z, x) \in \mathbb{C}_+ \times \mathbb{R}, \quad (4.71)$$

with $m_\pm(\cdot, x)$ the half-line Weyl–Titchmarsh functions of H . Relation (4.71) will be helpful in introducing $m_{\pm, r_0}(z, 0)$ for H_{r_0} . To this end we define

$$g_{r_0}(z, 0) = r_0(z) = \frac{i}{2z^{1/2}} \prod_{j \in \mathbb{N}} \left(\frac{z - b_j}{z - a_j} \right)^{1/2} = \int_{\mathfrak{E}} \frac{d\sigma_0(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+ \quad (4.72)$$

and write

$$\begin{aligned} -g_{r_0}(z, 0)^{-1} &= -r_0(z)^{-1} = m_{+, r_0}(z) - m_{-, r_0}(z) \\ &= 2iz^{1/2} \prod_{j \in \mathbb{N}} \left(\frac{z - a_j}{z - b_j} \right)^{1/2} \end{aligned} \quad (4.73)$$

$$= \text{Re}(-g(i)^{-1}) + \int_{\mathbb{R}} d\omega_{r_0}(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \quad (4.74)$$

and

$$\pm m_{\pm, r_0}(z) = \pm \text{Re}(m_{\pm, r_0}(i)) + \int_{\mathbb{R}} d\omega_{\pm, r_0}(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+. \quad (4.75)$$

Here $m_{\pm, r_0}(z) = m_{\pm, r_0}(z, 0)$ will be chosen next to represent the half-line Weyl–Titchmarsh functions of H_{r_0} . Because of $g_{r_0}(\cdot, 0) \in \mathcal{R}(\mathfrak{E})$, one infers

$$m_{+, r_0}(\lambda + i0) = \overline{m_{-, r_0}(\lambda + i0)} \text{ for a.e. } \lambda \in \mathfrak{E} \quad (4.76)$$

and hence,

$$-g_{r_0}(\lambda + i0, 0)^{-1} = \pm 2i \text{Im}(m_{\pm, r_0}(\lambda + i0)) \text{ for a.e. } \lambda \in \mathfrak{E}. \quad (4.77)$$

By (4.72)–(4.75), (4.77), (A.19), (B.49), (B.51), and the fact that any singular continuous component of a measure must be supported on an uncountable set, one concludes

$$\text{supp}(d\omega_{\pm, r_0, \text{ac}}) = \text{supp}(d\omega_{r_0, \text{ac}}) = \mathfrak{E}, \quad d\omega_{\pm, r_0, \text{ac}} = \frac{1}{2}d\omega_{r_0, \text{ac}}, \quad (4.78)$$

$$d\omega_{\pm, r_0, \text{sc}} = d\omega_{r_0, \text{sc}} = 0. \quad (4.79)$$

Moreover, since

$$\lim_{\varepsilon \downarrow 0} (-\varepsilon)g(-\varepsilon, 0) = \lim_{\varepsilon \downarrow 0} \varepsilon r_0(-\varepsilon) = \sigma_0(\{0\}) > 0 \quad (4.80)$$

by assumption (cf. Theorem 4.6), and the Herglotz property of $\pm m_{\pm, r_0}$ implies that

$$\lim_{\varepsilon \downarrow 0} (\pm \varepsilon)m_{\pm, r_0}(-\varepsilon) \geq 0, \quad (4.81)$$

one infers

$$\lim_{\varepsilon \downarrow 0} (-\varepsilon)g(-\varepsilon, 0)^{-1} = \lim_{\varepsilon \downarrow 0} (\pm \varepsilon)m_{\pm, r_0}(-\varepsilon) = 0 \quad (4.82)$$

and hence

$$\omega_{r_0, \text{pp}}(\{0\}) = \omega_{\pm, r_0, \text{pp}}(\{0\}) = 0. \quad (4.83)$$

Since by (4.72) and (B.52),

$$\text{supp}(\omega_{r_0, \text{pp}}) \cap (\mathbb{R} \setminus \{0\}) = \text{supp}(\omega_{\pm, r_0, \text{pp}}) \cap (\mathbb{R} \setminus \{0\}) = \emptyset, \quad (4.84)$$

one finally concludes

$$d\omega_{r_0, \text{pp}} = d\omega_{\pm, r_0, \text{pp}} = 0. \quad (4.85)$$

Next we introduce

$$\begin{aligned} h_{r_0}(z, 0) &= \frac{m_{-, r_0}(z)m_{+, r_0}(z)}{m_{-, r_0}(z) - m_{+, r_0}(z)} = g_{r_0}(z, 0)m_{-, r_0}(z)m_{+, r_0}(z) \\ &= \frac{i}{2z^{1/2}} \prod_{j \in \mathbb{N}} \left(\frac{z - b_j}{z - a_j} \right)^{1/2} m_{-, r_0}(z)m_{+, r_0}(z) \end{aligned} \quad (4.86)$$

$$= \text{Re}(h_{r_0}(i)) + \int_{\mathbb{R}} d\rho_0(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+. \quad (4.87)$$

A comparison of (4.73) and (4.86) then proves that $d\rho_0$ has no pure points on $(0, \infty)$. Moreover, since h_{r_0} and g_{r_0} are Herglotz, and the measure $d\sigma_0$ associated with g_{r_0} has a point mass at 0, considering the expression

$$\frac{\varepsilon h_{r_0}(-\varepsilon, 0)}{\varepsilon g_{r_0}(-\varepsilon, 0)} = m_{+, r_0}(-\varepsilon)m_{-, r_0}(-\varepsilon) \quad (4.88)$$

implies that

$$\lim_{\varepsilon \downarrow 0} m_{+, r_0}(-\varepsilon)m_{-, r_0}(-\varepsilon) \geq 0 \text{ exists finitely.} \quad (4.89)$$

In particular,

$$\rho_0(\{0\}) > 0 \text{ if and only if } \lim_{\varepsilon \downarrow 0} m_{+, r_0}(-\varepsilon)m_{-, r_0}(-\varepsilon) > 0. \quad (4.90)$$

Summing up, one concludes

$$\text{supp}(d\rho_0) = \text{supp}(d\rho_{0, \text{ac}}) = \mathfrak{E}, \quad \text{supp}(d\rho_{0, \text{pp}}) \subseteq \{0\}, \quad d\rho_{0, \text{sc}} = 0. \quad (4.91)$$

In order to apply Theorem 4.8 to m_{+,r_0} , one considers the function

$$\Phi(x) = \lim_{R \uparrow \infty} \int_0^R d\tilde{\omega}_+(\lambda) \frac{\sin(\lambda^{1/2}x)}{\lambda^{1/2}}, \quad x \geq 0, \quad (4.92)$$

where

$$d\tilde{\omega}_+(\lambda) = \pi^{-1} \lambda^{1/2} \left[\prod_{j \in \mathbb{N}} \left(\frac{\lambda - a_j}{\lambda - b_j} \right)^{1/2} \chi_{\mathfrak{E}}(\lambda) - 1 \right] d\lambda \quad (4.93)$$

with $\chi_{\mathfrak{E}}$ the characteristic function of the set \mathfrak{E} . Splitting the integral in (4.92) over the intervals $[0, b_1]$ and $[b_1, \infty)$,

$$\Phi_1(x) = \int_0^{b_1} d\tilde{\omega}_+(\lambda) \frac{\sin(\lambda^{1/2}x)}{\lambda^{1/2}}, \quad \Phi_2(x) = \lim_{R \uparrow \infty} \int_{b_1}^R d\tilde{\omega}_+(\lambda) \frac{\sin(\lambda^{1/2}x)}{\lambda^{1/2}}, \quad x \geq 0, \quad (4.94)$$

clearly,

$$\Phi_1 \in C^\infty((0, \infty)), \quad \Phi_1^{(m)} \in L^\infty([0, \infty); dx) \text{ for all } m \in \mathbb{N}_0, \quad \lim_{x \downarrow 0} \Phi_1(x) = 0. \quad (4.95)$$

Applying Lemma 4.10 to Φ_2 then yields all the properties in (4.95) for Φ_2 as well. Thus, Theorem 4.8 shows that m_{+,r_0} is the half-line Weyl–Titchmarsh function of a Schrödinger operator H_{+,r_0} in $L^2((0, \infty); dx)$ with a Dirichlet boundary condition at $x = 0$ and a real-valued potential coefficient V_{+,r_0} satisfying

$$V_{+,r_0}^{(m)} \in L^1([0, R]; dx) \text{ for all } m \in \mathbb{N}_0 \text{ and all } R > 0. \quad (4.96)$$

Analogous considerations then prove that m_{-,r_0} is the half-line Weyl–Titchmarsh function of a Schrödinger operator H_{-,r_0} in $L^2((-\infty, 0]; dx)$ with a Dirichlet boundary condition at $x = 0$ and a real-valued potential coefficient V_{-,r_0} satisfying

$$V_{-,r_0}^{(m)} \in L^1([-R, 0]; dx) \text{ for all } m \in \mathbb{N}_0 \text{ and all } R > 0. \quad (4.97)$$

Given V_{\pm,r_0} we next introduce

$$V_{r_0}(x) = \begin{cases} V_{+,r_0}(x), & x > 0, \\ V_{-,r_0}(x), & x < 0, \end{cases} \quad (4.98)$$

and note that V_{r_0} satisfies the properties in (4.68). Finally, introducing the differential expression

$$L_{r_0} = -d^2/dx^2 + V_{r_0}(x), \quad x \in \mathbb{R}, \quad (4.99)$$

and denoting by H_{r_0} the corresponding self-adjoint realization of L_{r_0} in $L^2(\mathbb{R}; dx)$,

$$H_{r_0}f = L_{r_0}f, \quad f \in \text{dom}(H_{r_0}) = \{g \in L^2(\mathbb{R}; dx) \mid g, g' \in AC_{\text{loc}}(\mathbb{R}); L_{r_0}g \in L^2(\mathbb{R}; dx)\}, \quad (4.100)$$

a study of the trace measure of H_{r_0} ,

$$d\Omega_{r_0}^{\text{tr}}(\lambda) = d\sigma_0(\lambda) + d\rho_0(\lambda) \quad (4.101)$$

then yields the spectral properties of H_{r_0} in (4.69). \square

For simplicity we singled out the eigenvalue 0 in H_{r_0} (which is of course correlated with the point mass at 0 of $d\sigma_0$ in Lemma 4.4). A similar construction leads to an eigenvalue of a reflectionless Schrödinger operator at a point $\lambda_0 > 0$.

For an entirely different construction of reflectionless tridiagonal matrices (Jacobi operators) on the lattice \mathbb{Z} with empty singular continuous spectra but possibly

countably many accumulation points in the set of eigenvalues as well as in the set of boundary points of intervals of absolutely continuous spectrum, we refer to [27].

APPENDIX A. ESSENTIAL CLOSURES OF SETS AND ESSENTIAL SUPPORTS OF MEASURES

The following material on essential closures of essential supports of absolutely continuous measures is probably well-known, but we found no comprehensive treatment in the literature and hence decided to collect the relevant facts in this appendix.

For basic facts on measures on \mathbb{R} relevant to this appendix we refer, for instance, to [2]–[4], [16, p. 179], [18], [29]–[32], [59, Sect. V.12], [63, p. 140–141], [69], [71]. All measures in this appendix will be assumed to be nonnegative without explicitly stressing this fact again.

Since Borel and Borel–Stieltjes measures are incomplete (i.e., not any subset of a set of measure zero is measurable) we will enlarge the Borel σ -algebra to obtain the complete Lebesgue and Lebesgue–Stieltjes measures. We recall the standard Lebesgue decomposition of a measure $d\mu$ on \mathbb{R} with respect to Lebesgue measure dx on \mathbb{R} ,

$$d\mu = d\mu_{ac} + d\mu_s = d\mu_{ac} + d\mu_{sc} + d\mu_{pp}, \quad (\text{A.1})$$

$$d\mu_{ac} = f dx, \quad 0 \leq f \in L^1_{loc}(\mathbb{R}; dx), \quad (\text{A.2})$$

where $d\mu_{ac}$, $d\mu_{sc}$, and $d\mu_{pp}$ denote the absolutely continuous, singularly continuous, and pure point parts of $d\mu$, respectively. The Lebesgue measure of a Lebesgue measurable set $\Omega \subset \mathbb{R}$ will be denoted by $|\Omega|$.

In the following, the Lebesgue measure of a set S will be denoted by $|S|$ and all sets whose μ -measure or Lebesgue measure is considered are always assumed to be Lebesgue–Stieltjes or Lebesgue measurable, etc.

Definition A.1. Let $d\mu$ be a Lebesgue–Stieltjes measure and suppose S and S' are μ -measurable.

- (i) S is called a *support* of $d\mu$ if $\mu(\mathbb{R} \setminus S) = 0$.
- (ii) The smallest closed support of $d\mu$ is called the *topological support* of $d\mu$ and denoted by $\text{supp}(d\mu)$.
- (iii) S is called an *essential* (or *minimal*) *support* of $d\mu$ (relative to Lebesgue measure dx on \mathbb{R}) if $\mu(\mathbb{R} \setminus S) = 0$, and $S' \subseteq S$ with $S' \mid \cdot \mid$ -measurable, $\mu(S') = 0$ imply $|S'| = 0$.

Remark A.2. Item (iii) in Definition A.1 is equivalent to

- (iii') S is called an *essential* (or *minimal*) *support* of $d\mu$ (relative to Lebesgue measure dx on \mathbb{R}) if $\mu(\mathbb{R} \setminus S) = 0$, and $S' \subseteq S$, $\mu(\mathbb{R} \setminus S') = 0$ imply $|S \setminus S'| = 0$.

Lemma A.3 ([29]). Let $S, S' \subseteq \mathbb{R}$ be μ - and $|\cdot|$ -measurable. Define the relation \sim by $S \sim S'$ if

$$\mu(S \Delta S') = |S \Delta S'| = 0 \quad (\text{A.3})$$

(where $S \Delta S' = (S \setminus S') \cup (S' \setminus S)$). Then \sim is an equivalence relation. Moreover, the set of all essential supports of $d\mu$ is an equivalence class under \sim .

Example A.4. Let $d\mu_{pp}$ be a pure point measure and

$$\mu_{pp}(\{x\}) = \begin{cases} c(x) > 0, & x \in [0, 1] \cap \mathbb{Q}, \\ 0, & x \in [0, 1] \setminus \mathbb{Q} \text{ or } |x| > 1. \end{cases} \quad (\text{A.4})$$

Then,

$$\text{supp}(d\mu_{\text{pp}}) = [0, 1]. \quad (\text{A.5})$$

However, since $[0, 1] \cap \mathbb{Q}$ is an essential support of $d\mu_{\text{pp}}$ and since $|[0, 1] \cap \mathbb{Q}| = 0$, also

$$|S_{\mu_{\text{pp}}}| = 0 \quad (\text{A.6})$$

for any other essential support $S_{\mu_{\text{pp}}}$ of $d\mu_{\text{pp}}$.

Remark A.5. If $d\mu = d\mu_{\text{ac}}$, then $|S\Delta S'| = 0$ implies $\mu(S\Delta S') = 0$ and hence any two essential supports of $d\mu$ differ at most by sets of Lebesgue measure zero. Indeed, one can use the following,

$$S_1 = (S_2 \cap S_1) \cup (S_1 \setminus S_2), \quad S_2 = (S_1 \cap S_2) \cup (S_2 \setminus S_1), \quad (\text{A.7})$$

$$S_1 \cup (S_2 \setminus S_1) = S_2 \cup (S_1 \setminus S_2) \quad (\text{A.8})$$

for any subsets $S_j \subset \mathbb{R}$, $j = 1, 2$.

Definition A.6. Let $A \subset \mathbb{R}$ be Lebesgue measurable. Then the *essential closure* \overline{A}^e of A is defined as

$$\overline{A}^e = \{x \in \mathbb{R} \mid \text{for all } \varepsilon > 0: |(x - \varepsilon, x + \varepsilon) \cap A| > 0\}. \quad (\text{A.9})$$

The following is an immediate consequence of Definition A.6.

Lemma A.7. Let $A, B, C \subset \mathbb{R}$ be Lebesgue measurable. Then,

$$(i) \text{ If } A \subseteq B \text{ then } \overline{A}^e \subseteq \overline{B}^e. \quad (\text{A.10})$$

$$(ii) \text{ If } |A| = 0 \text{ then } \overline{A}^e = \emptyset. \quad (\text{A.11})$$

$$(iii) \text{ If } A = B \cup C \text{ with } |C| = 0, \text{ then } \overline{A}^e = \overline{B}^e. \quad (\text{A.12})$$

Example A.8.

(i) Consider $d\mu_{\text{pp}}$ in Example A.4. Let $S_{\mu_{\text{pp}}}$ be any essential support of $d\mu_{\text{pp}}$. Then $\overline{S_{\mu_{\text{pp}}}}^e = \emptyset$ by (A.11).

(ii) Consider $A = [0, 1] \cup \{2\}$. Then $\overline{A}^e = [0, 1]$.

Lemma A.9. Let $A \subseteq \mathbb{R}$ be Lebesgue measurable. Then,

$$(i) \quad \overline{A}^e \text{ is a closed set.} \quad (\text{A.13})$$

$$(ii) \quad \overline{A}^e \subseteq \overline{A}. \quad (\text{A.14})$$

$$(iii) \quad \overline{(\overline{A}^e)}^e \subseteq \overline{A}^e. \quad (\text{A.15})$$

Proof. (i) We will show that the set

$$\mathbb{R} \setminus \overline{A}^e = \{x \in \mathbb{R} \mid \text{there is an } \varepsilon_0 > 0 \text{ such that } |(x - \varepsilon_0, x + \varepsilon_0) \cap A| = 0\} \quad (\text{A.16})$$

is open. Pick $x_0 \in \mathbb{R} \setminus \overline{A}^e$, then there is an $\varepsilon_0 > 0$ such that $|(x_0 - \varepsilon_0, x_0 + \varepsilon_0) \cap A| = 0$. Consider $x_1 \in (x_0 - (\varepsilon_0/4), x_0 + (\varepsilon_0/4))$ and the open ball $S(x_1; \varepsilon_0/4)$ centered at x_1 with radius $\varepsilon_0/4$. Then,

$$|S(x_1; \varepsilon_0/4) \cap A| \leq |(x_0 - \varepsilon_0, x_0 + \varepsilon_0) \cap A| = 0 \quad (\text{A.17})$$

and hence $x_1 \in \mathbb{R} \setminus \overline{A}^e$ and $S(x_0; \varepsilon_0/4) \subseteq \mathbb{R} \setminus \overline{A}^e$. Thus, $\mathbb{R} \setminus \overline{A}^e$ is open.

(ii) Let $x \in \overline{A}^e$. Then for all $\varepsilon > 0$, $|(x - \varepsilon, x + \varepsilon) \cap A| > 0$. Choose $\varepsilon_n = 1/n$, $n \in \mathbb{N}$, then $(x - \varepsilon_n, x + \varepsilon_n) \cap A \neq \emptyset$ and we may choose an $x_n \in (x - \varepsilon_n, x + \varepsilon_n) \cap A$. Since $x_n \rightarrow x$ as $n \rightarrow \infty$, $x \in \overline{A}$ and hence $\overline{A}^e \subseteq \overline{A}$.

(iii) By (ii), $\overline{A}^e \subseteq \overline{A}$. Hence, $\overline{(\overline{A}^e)}^e \subseteq \overline{\overline{A}^e} = \overline{A}^e$ since \overline{A}^e is closed by (i). \square

Lemma A.10. *Let $d\mu = d\mu_{ac}$ and S_1 and S_2 be essential supports of $d\mu$. Then,*

$$\overline{S_1}^e = \overline{S_2}^e. \quad (\text{A.18})$$

Proof. Since $|S_1 \setminus S_2| = |S_2 \setminus S_1| = 0$, (A.18) follows from (A.7) and (A.11). \square

Actually, one can improve on Lemma A.10:

Lemma A.11. *Let $d\mu = d\mu_{ac} = f dx$, $0 \leq f \in L^1_{loc}(\mathbb{R})$. If S is any essential support of $d\mu$, then,*

$$\overline{S}^e = \overline{\{x \in \mathbb{R} \mid f(x) > 0\}}^e = \text{supp}(d\mu). \quad (\text{A.19})$$

Proof. Since $\{x \in \mathbb{R} \mid f(x) > 0\}$ is an essential support of $d\mu$, it suffices to prove

$$\overline{\{x \in \mathbb{R} \mid f(x) > 0\}}^e = \text{supp}(d\mu). \quad (\text{A.20})$$

We denote $U = \mathbb{R} \setminus \text{supp}(d\mu)$. Then U is the largest open set that satisfies $\mu(U) = 0$.

Next, let $U' = \mathbb{R} \setminus \overline{\{x \in \mathbb{R} \mid f(x) > 0\}}^e$. By Lemma A.9 (i), U' is open.

“ \supseteq ”: Let $x \in U'$. Then there is an $\varepsilon_0 > 0$ such that

$$|(x - \varepsilon_0, x + \varepsilon_0) \cap \{y \in \mathbb{R} \mid f(y) > 0\}| = 0 \quad (\text{A.21})$$

(cf. (A.16)). Hence,

$$f = 0 \mid \cdot \mid\text{-a.e. on } (x - \varepsilon_0, x + \varepsilon_0) \quad (\text{A.22})$$

and thus, $\mu((x - \varepsilon_0, x + \varepsilon_0)) = 0$. Next one covers U' with open intervals of this form to arrive at $\mu(U') = 0$. Since U is the largest open set satisfying $\mu(U) = 0$, one infers $U' \subseteq U$ and hence

$$\overline{\{x \in \mathbb{R} \mid f(x) > 0\}}^e \supseteq \text{supp}(d\mu). \quad (\text{A.23})$$

“ \subseteq ”: Suppose $x \in U$. Since $\text{supp}(d\mu)$ is closed, there is an $\varepsilon_0 > 0$ such that $(x - \varepsilon_0, x + \varepsilon_0) \cap \text{supp}(d\mu) = \emptyset$. Thus, $\mu((x - \varepsilon, x + \varepsilon)) = 0$ for all $0 \leq \varepsilon \leq \varepsilon_0$. Actually, $\mu(B) = 0$ for all μ -measurable $B \subseteq (x - \varepsilon_0, x + \varepsilon_0)$ and hence $f = 0 \mid \cdot \mid\text{-a.e. on } (x - \varepsilon_0, x + \varepsilon_0)$. Thus,

$$|(x - \varepsilon, x + \varepsilon) \cap \{y \in \mathbb{R} \mid f(y) > 0\}| = 0 \text{ for all } 0 \leq \varepsilon \leq \varepsilon_0 \quad (\text{A.24})$$

and one obtains $x \in U'$. Thus, $U' \supseteq U$ and hence

$$\overline{\{x \in \mathbb{R} \mid f(x) > 0\}}^e \subseteq \text{supp}(d\mu). \quad (\text{A.25})$$

\square

We remark that a result of the type (A.19) has been noted in [7, Corollary 11.11] in the context of general ordinary differential operators and their associated Weyl–Titchmarsh matrices. In this connection we also refer to [77, p. 301] for a corresponding result in connection with Herglotz functions and their associated measures.

APPENDIX B. HERGLOTZ FUNCTIONS AND WEYL–TITCHMARSH THEORY FOR SCHRÖDINGER OPERATORS IN A NUTSHELL

The material in this appendix is well-known, but since we use it at various places in this paper, we thought it worthwhile to collect it in an appendix.

Definition B.1. Let $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \text{Im}(z) \gtrless 0\}$. $m : \mathbb{C}_+ \rightarrow \mathbb{C}$ is called a *Herglotz function* (or *Nevanlinna* or *Pick function*) if m is analytic on \mathbb{C}_+ and $m(\mathbb{C}_+) \subseteq \mathbb{C}_+$.

One then extends m to \mathbb{C}_- by reflection, that is, one defines

$$m(z) = \overline{m(\bar{z})}, \quad z \in \mathbb{C}_-. \quad (\text{B.1})$$

Of course, generally, (B.1) does not represent the analytic continuation of $m|_{\mathbb{C}_+}$ into \mathbb{C}_- .

The fundamental result on Herglotz functions and their representations on Borel transforms, in part due to Fatou, Herglotz, Luzin, Nevanlinna, Plessner, Privalov, de la Vallée Poussin, Riesz, and others, then reads as follows.

Theorem B.2. ([1], Sect. 69, [3], [19], Chs. II, IV, [42], [43], Ch. 6, [62], Chs. II, IV, [66], Ch. 5).

Let m be a Herglotz function. Then,

- (i) $m(z)$ has finite normal limits $m(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} m(\lambda \pm i\varepsilon)$ for a.e. $\lambda \in \mathbb{R}$.
- (ii) Suppose $m(z)$ has a zero normal limit on a subset of \mathbb{R} having positive Lebesgue measure. Then $m \equiv 0$.
- (iii) There exists a nonnegative measure $d\omega$ on \mathbb{R} satisfying

$$\int_{\mathbb{R}} \frac{d\omega(\lambda)}{1 + \lambda^2} < \infty \quad (\text{B.2})$$

such that the Nevanlinna, respectively, Riesz-Herglotz representation

$$m(z) = c + dz + \int_{\mathbb{R}} d\omega(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \quad (\text{B.3})$$

$$c = \operatorname{Re}[m(i)], \quad d = \lim_{\eta \uparrow \infty} m(i\eta)/(i\eta) \geq 0$$

holds. Conversely, any function m of the type (B.3) is a Herglotz function.

- (iv) Let $(\lambda_1, \lambda_2) \subset \mathbb{R}$, then the Stieltjes inversion formula for $d\omega$ reads

$$\frac{1}{2}\omega(\{\lambda_1\}) + \frac{1}{2}\omega(\{\lambda_2\}) + \omega((\lambda_1, \lambda_2)) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \int_{\lambda_1}^{\lambda_2} d\lambda \operatorname{Im}(m(\lambda + i\varepsilon)). \quad (\text{B.4})$$

- (v) The absolutely continuous (ac) part $d\omega_{ac}$ of $d\omega$ with respect to Lebesgue measure $d\lambda$ on \mathbb{R} is given by

$$d\omega_{ac}(\lambda) = \pi^{-1} \operatorname{Im}[m(\lambda + i0)] d\lambda. \quad (\text{B.5})$$

- (vi) Local singularities of m and m^{-1} are necessarily real and at most of first order in the sense that

$$\lim_{\varepsilon \downarrow 0} (-i\varepsilon) m(\lambda + i\varepsilon) \geq 0, \quad \lambda \in \mathbb{R}, \quad (\text{B.6})$$

$$\lim_{\varepsilon \downarrow 0} (i\varepsilon) m(\lambda + i\varepsilon)^{-1} \geq 0, \quad \lambda \in \mathbb{R}. \quad (\text{B.7})$$

Next, we denote by

$$d\omega = d\omega_{ac} + d\omega_{sc} + d\omega_{pp} \quad (\text{B.8})$$

the decomposition of $d\omega$ into its absolutely continuous (ac), singularly continuous (sc), and pure point (pp) parts with respect to Lebesgue measure on \mathbb{R} .

Theorem B.3 ([3], [42], [68], [69]). Let m be a Herglotz function with representation (B.3). Then,

(i)

$$d = 0 \text{ and } \int_{\mathbb{R}} \frac{d\omega(\lambda)}{1 + |\lambda|^s} < \infty \text{ for some } s \in (0, 2) \quad (\text{B.9})$$

$$\text{if and only if } \int_1^\infty d\eta \eta^{-s} \operatorname{Im}[m(i\eta)] < \infty.$$

(ii) Let $(\lambda_1, \lambda_2) \subset \mathbb{R}$, $\eta_1 > 0$. Then there is a constant $C(\lambda_1, \lambda_2, \eta_1) > 0$ such that

$$\eta |m(\lambda + i\eta)| \leq C(\lambda_1, \lambda_2, \eta_1), \quad (\lambda, \eta) \in [\lambda_1, \lambda_2] \times (0, \eta_1). \quad (\text{B.10})$$

(iii)

$$\sup_{\eta > 0} \eta |m(i\eta)| < \infty \text{ if and only if } m(z) = \int_{\mathbb{R}} \frac{d\omega(\lambda)}{\lambda - z} \text{ and } \int_{\mathbb{R}} d\omega(\lambda) < \infty. \quad (\text{B.11})$$

In this case,

$$\int_{\mathbb{R}} d\omega(\lambda) = \sup_{\eta > 0} \eta |m(i\eta)| = -i \lim_{\eta \uparrow \infty} \eta m(i\eta). \quad (\text{B.12})$$

(iv) For all $\lambda \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \operatorname{Re}[m(\lambda + i\varepsilon)] = 0, \quad (\text{B.13})$$

$$\omega(\{\lambda\}) = \lim_{\varepsilon \downarrow 0} \varepsilon \operatorname{Im}[m(\lambda + i\varepsilon)] = -i \lim_{\varepsilon \downarrow 0} \varepsilon m(\lambda + i\varepsilon). \quad (\text{B.14})$$

(v) Let $L > 0$ and suppose $0 \leq \operatorname{Im}[m(z)] \leq L$ for all $z \in \mathbb{C}_+$. Then $d = 0$, $d\omega$ is purely absolutely continuous, $d\omega = d\omega_{ac}$, and

$$0 \leq \frac{d\omega(\lambda)}{d\lambda} = \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im}[m(\lambda + i\varepsilon)] \leq \pi^{-1} L \text{ for a.e. } \lambda \in \mathbb{R}. \quad (\text{B.15})$$

(vi) Let $p \in (1, \infty)$, $[\lambda_3, \lambda_4] \subset (\lambda_1, \lambda_2)$, $[\lambda_1, \lambda_2] \subset (\lambda_5, \lambda_6)$. If

$$\sup_{0 < \varepsilon < 1} \int_{\lambda_1}^{\lambda_2} d\lambda |\operatorname{Im}[m(\lambda + i\varepsilon)]|^p < \infty, \quad (\text{B.16})$$

then $d\omega = d\omega_{ac}$ is purely absolutely continuous on (λ_1, λ_2) , $\frac{d\omega_{ac}}{d\lambda} \in L^p((\lambda_1, \lambda_2); d\lambda)$, and

$$\lim_{\varepsilon \downarrow 0} \left\| \pi^{-1} \operatorname{Im}[m(\cdot + i\varepsilon)] - \frac{d\omega_{ac}}{d\lambda} \right\|_{L^p((\lambda_3, \lambda_4); d\lambda)} = 0. \quad (\text{B.17})$$

Conversely, if $d\omega$ is purely absolutely continuous on (λ_5, λ_6) , and if $\frac{d\omega_{ac}}{d\lambda} \in L^p((\lambda_5, \lambda_6); d\lambda)$, then (B.16) holds.(vii) Let $(\lambda_1, \lambda_2) \subset \mathbb{R}$. Then a local version of Wiener's theorem reads for $p \in (1, \infty)$,

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \varepsilon^{p-1} \int_{\lambda_1}^{\lambda_2} d\lambda |\operatorname{Im}[m(\lambda + i\varepsilon)]|^p \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(p - \frac{1}{2})}{\Gamma(p)} \left[\frac{1}{2} \omega(\{\lambda_1\})^p + \frac{1}{2} \omega(\{\lambda_2\})^p + \sum_{\lambda \in (\lambda_1, \lambda_2)} \omega(\{\lambda\})^p \right]. \end{aligned} \quad (\text{B.18})$$

Moreover, for $0 < p < 1$,

$$\lim_{\varepsilon \downarrow 0} \int_{\lambda_1}^{\lambda_2} d\lambda |\pi^{-1} \operatorname{Im}[m(\lambda + i\varepsilon)]|^p = \int_{\lambda_1}^{\lambda_2} d\lambda \left| \frac{d\omega_{ac}(\lambda)}{d\lambda} \right|^p. \quad (\text{B.19})$$

Together with m , $\ln(m)$ is a Herglotz function. Moreover, since

$$0 \leq \operatorname{Im}[\ln(m(z))] = \arg[m(z)] \leq \pi, \quad z \in \mathbb{C}_+, \quad (\text{B.20})$$

the measure $d\widehat{\omega}$ in the representation (B.3) of $\ln(m)$, that is, in the exponential representation of m , is purely absolutely continuous by Theorem B.3(v), $d\widehat{\omega}(\lambda) = \xi(\lambda)d\lambda$ for some $0 \leq \xi \leq 1$. These exponential representations have been studied in great detail by Aronszajn and Donoghue [3], [4] and we record a few of their properties below.

Theorem B.4 ([3], [4]). *Suppose $m(z)$ is a Herglotz function with representation (B.3). Then*

(i) *There exists a $\xi \in L^\infty(\mathbb{R})$, $0 \leq \xi \leq 1$ a.e., such that*

$$\begin{aligned} \ln(m(z)) &= k + \int_{\mathbb{R}} d\lambda \xi(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \\ k &= \operatorname{Re}[\ln(m(i))], \end{aligned} \quad (\text{B.21})$$

where

$$\xi(\lambda) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im}[\ln(m(\lambda + i\varepsilon))] \quad \text{a.e.} \quad (\text{B.22})$$

(ii) *Let $\ell_1, \ell_2 \in \mathbb{N}$ and $d = 0$ in (B.3). Then*

$$\begin{aligned} &\int_{-\infty}^0 d\lambda \xi(\lambda) \frac{|\lambda|^{\ell_1}}{1 + \lambda^2} + \int_0^\infty d\lambda \xi(\lambda) \frac{|\lambda|^{\ell_2}}{1 + \lambda^2} < \infty \\ &\text{if and only if } \int_{-\infty}^0 d\omega(\lambda) \frac{|\lambda|^{\ell_1}}{1 + \lambda^2} + \int_0^\infty d\omega(\lambda) \frac{|\lambda|^{\ell_2}}{1 + \lambda^2} < \infty \\ &\text{and } \lim_{z \rightarrow i\infty} m(z) = c - \int_{\mathbb{R}} d\omega(\lambda) \frac{\lambda}{1 + \lambda^2} > 0. \end{aligned} \quad (\text{B.23})$$

(iii)

$$\begin{aligned} &\xi(\lambda) = 0 \text{ for } \lambda < 0 \text{ if and only if} \\ &d = 0, \quad [0, \infty) \text{ is a support for } \omega \text{ (i.e., } \omega((-\infty, 0)) = 0), \\ &\int_0^\infty \frac{d\omega(\lambda)}{1 + \lambda} < \infty, \text{ and } c \geq \int_0^\infty d\omega(\lambda) \frac{\lambda}{1 + \lambda^2}. \end{aligned} \quad (\text{B.24})$$

In this case

$$\lim_{\lambda \downarrow -\infty} m(\lambda) = c - \int_0^\infty d\omega(\lambda') \frac{\lambda'}{1 + \lambda'^2} \quad (\text{B.25})$$

and

$$c > \int_0^\infty d\omega(\lambda) \frac{\lambda}{1 + \lambda^2} \text{ if and only if } \int_0^\infty \frac{d\lambda \xi(\lambda)}{1 + \lambda} < \infty. \quad (\text{B.26})$$

(iv) *Let $(\lambda_1, \lambda_2) \subset \mathbb{R}$ and suppose $0 \leq A \leq \xi(\lambda) \leq B \leq 1$ for a.e. $\lambda \in (\lambda_1, \lambda_2)$ with $(B - A) < 1$. Then ω is purely absolutely continuous in (λ_1, λ_2) and $\frac{d\omega}{d\lambda} \in L^p((\lambda_3, \lambda_4); d\lambda)$ for $[\lambda_3, \lambda_4] \subset (\lambda_1, \lambda_2)$ and all $p < (B - A)^{-1}$.*

(v) *The measure ω is purely singular, $\omega = \omega_s$, $\omega_{ac} = 0$ if and only if ξ equals the characteristic function of a measurable subset $A \subseteq \mathbb{R}$, that is, $\xi = \chi_A$.*

While Theorems B.3(v), (vi) and B.4(iv) describe necessary conditions for $d\omega$ to be purely absolutely continuous on an interval (λ_1, λ_2) , we need to appeal to a stronger result in Section 3. Next, we recall some basic facts from [21, Chs. 10, 11], [23, Ch. II], [43, Ch. VI], [66, Ch. 5] to prepare the ground for this result.

Definition B.5. Let $F: \mathbb{C}_+ \rightarrow \mathbb{C}$ be analytic and $C \in \mathbb{C}$ a constant satisfying $|C| = 1$.

(i) F is called an *outer function* on \mathbb{C}_+ if

$$F(z) = C \exp \left(\frac{1}{\pi i} \int_{\mathbb{R}} dx \log(K(x)) \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) \right), \quad z \in \mathbb{C}_+, \quad (\text{B.27})$$

where $K > 0$ a.e. on \mathbb{R} and $\int_{\mathbb{R}} dx |\log(K(x))| (1+x^2)^{-1} < \infty$. In this case, $K(x) = |F(x+i0)|$ for a.e. $x \in \mathbb{R}$.

(ii) F is called a *Blaschke product* if

$$F(z) = C \left(\frac{z-i}{z+i} \right)^n \prod_{j \in J} \frac{|z_j^2+1|}{z_j^2+1} \frac{z-z_j}{z-\bar{z}_j}, \quad z \in \mathbb{C}_+, \quad \{z_j\}_{j \in J} \in \mathbb{C}_+ \setminus \{i\}, \quad (\text{B.28})$$

where $n \in \mathbb{N}_0$, and $J \subseteq \mathbb{N}$ is a (possibly empty) index set. One then has

$$\sum_{j \in J} \frac{y_j}{x_j^2 + (y_j+1)^2} < \infty, \quad z_j = x_j + iy_j, \quad j \in J. \quad (\text{B.29})$$

(iii) F is called a *singular inner function* on \mathbb{C}_+ if

$$F(z) = C e^{iaz} \exp \left(i \int_{\mathbb{R}} d\mu_s(x) \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) \right), \quad z \in \mathbb{C}_+, \quad (\text{B.30})$$

where $a \in (0, \infty)$ and $d\mu_s$ is a nonnegative singular measure on \mathbb{R} satisfying $\int_{\mathbb{R}} d\mu_s(x) (1+x^2)^{-1} < \infty$.

(iv) F is called an *inner function* if $F(z) = B_F(z) S_F(z)$, $z \in \mathbb{C}_+$, where B_F is a Blaschke product and S_F is a singular inner function. In this case, the factors B_F and S_F are unique up to multiplicative unimodular constants.

We also briefly recall the Nevanlinna, Smirnov, and Hardy classes associated with \mathbb{C}_+ :

Definition B.6. (i) The *Nevanlinna class* $N(\mathbb{C}_+)$ is defined as the union of the identically vanishing function on \mathbb{C}_+ and the set of analytic functions $F: \mathbb{C}_+ \rightarrow \mathbb{C}$, $F \not\equiv 0$, such that

$$F(z) = I_F(z) O_F(z) / S_F(z), \quad z \in \mathbb{C}_+, \quad (\text{B.31})$$

where I_F is an inner, O_F is an outer, and S_F is a singular inner function on \mathbb{C}_+ .

(ii) The *Smirnov class* $N_+(\mathbb{C}_+)$ is defined as the union of the identically vanishing function on \mathbb{C}_+ and the set of analytic functions $F: \mathbb{C}_+ \rightarrow \mathbb{C}$, $F \not\equiv 0$, such that

$$F(z) = I_F(z) O_F(z), \quad z \in \mathbb{C}_+, \quad (\text{B.32})$$

where I_F is an inner and O_F is an outer function on \mathbb{C}_+ .

(iii) The Hardy spaces $H^p(\mathbb{C}_+)$, $p \in (0, \infty)$ are defined by

$$H^p(\mathbb{C}_+) = \left\{ F: \mathbb{C}_+ \rightarrow \mathbb{C} \text{ analytic} \mid \|F\|_{H^p(\mathbb{C}_+)}^p = \sup_{y>0} \int_{\mathbb{R}} dx |F(x+iy)|^p < \infty \right\}. \quad (\text{B.33})$$

(iv) The Hardy spaces $H^p(\mathbb{R})$, $p \in (0, \infty)$ are defined by

$$H^p(\mathbb{R}) = \left\{ f \in L^p(\mathbb{R}; dx) \mid \text{there exists an } F \in H^p(\mathbb{C}_+) \text{ such that} \right. \\ \left. \text{for a.e. } x \in \mathbb{R}, \lim_{z \rightarrow x} F(z) = f(x) \text{ nontangentially} \right\} \quad (\text{B.34})$$

with $\|f\|_{H^p(\mathbb{R})}^p = \|f\|_{L^p(\mathbb{R}; dx)}^p$.

Of course, $\|\cdot\|_{H^p(\mathbb{C}_+)}$ and $\|\cdot\|_{H^p(\mathbb{R})}$ are norms only for $p \geq 1$.

Moreover, $F \in N(\mathbb{C}_+)$ if and only if F is of the form $F = G/H$, where G and H are analytic and bounded on \mathbb{C}_+ , and H is nonvanishing on \mathbb{C}_+ . In addition, $F \in N_+(\mathbb{C}_+)$ if and only if F is of the form $F = H_F/O_F$, where H_F and O_F are analytic and bounded by 1 on \mathbb{C}_+ and O_F is outer on \mathbb{C}_+ . In particular, $N_+(\mathbb{C}_+)$ is the smallest algebra that contains all inner and outer functions on \mathbb{C}_+ . One has the (strict) inclusions

$$H^p(\mathbb{C}_+) \subset N_+(\mathbb{C}_+) \subset N(\mathbb{C}_+), \quad p > 0. \quad (\text{B.35})$$

Functions in $N(\mathbb{C}_+)$ which are not identically vanishing have a.e. nontangential boundary values on \mathbb{R} which cannot vanish on a set of positive Lebesgue measure.

Analogous facts hold with \mathbb{C}_+ replaced by \mathbb{C}_- .

One verifies that every Herglotz function F is an outer function on \mathbb{C}_+ and hence sums and differences of Herglotz functions lie in $N_+(\mathbb{C}_+)$.

For subsequent purpose in Theorem B.7 below we also need to make the connection with the real Hardy space $\widetilde{H}^1(\mathbb{R})$ in the sense of Stein and Weiss defined as follows (see, e.g., [76, Sect. III.5.12]):

$$\begin{aligned} \widetilde{H}^1(\mathbb{R}) &= \{f \in L^1(\mathbb{R}) \mid \text{there exist } F_j \in H^p(\mathbb{C}_+) \text{ such that } f = f_1 + \overline{f_2}, \\ &\quad \text{where } f_j(x) = \lim_{\varepsilon \downarrow 0} F_j(x + i\varepsilon) \text{ for a.e. } x \in \mathbb{R}, j = 1, 2\}. \end{aligned} \quad (\text{B.36})$$

Moreover, denoting by \mathcal{H} the Hilbert transform on $L^1(\mathbb{R}; dx)$,

$$(\mathcal{H}f)(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|t-x| \geq \varepsilon} dt \frac{f(t)}{t-x} \text{ for a.e. } x \in \mathbb{R}, \quad f \in L^1(\mathbb{R}; dx), \quad (\text{B.37})$$

then

$$f \in \widetilde{H}^1(\mathbb{R}) \text{ if and only if } f \in L^1(\mathbb{R}; dx) \text{ and } \mathcal{H}f \in L^1(\mathbb{R}; dx) \quad (\text{B.38})$$

and

$$f \in \widetilde{H}^1(\mathbb{R}) \text{ implies } \int_{\mathbb{R}} dx f(x) = 0. \quad (\text{B.39})$$

Next, we mention a result due to Zinsmeister [81] which goes beyond Theorems B.3(v), (vi) and B.4(iv). For this purpose we now introduce the space $\widetilde{H}^1(K)$, where $K \subset \mathbb{R}$ is compact and of positive Lebesgue measure, $|K| > 0$, as follows: Let $\mathcal{M}_{\mathbb{C}}(\mathbb{R})$ denote the space of complex-valued (and hence finite) Borel measures on \mathbb{R} . Denote by $d\nu \in \mathcal{M}_{\mathbb{C}}$ a complex-valued measure supported on K normalized by

$$\int_K d\nu(x) = 0 \quad (\text{B.40})$$

and introduce the function

$$F(z) = \int_K \frac{d\nu(x)}{x-z}, \quad z \in \mathbb{C} \setminus K. \quad (\text{B.41})$$

One then verifies that

$$F|_{\mathbb{C}_{\pm}} \in H^p(\mathbb{C}_{\pm}) \text{ for all } p \in (1/2, 1) \quad (\text{B.42})$$

and, of course, F has nontangential limits

$$F_{\pm}(x) = \lim_{z \in \mathbb{C}_{\pm}, z \rightarrow x} F(z) \text{ for a.e. } x \in K. \quad (\text{B.43})$$

The space $\widetilde{H}^1(K)$ is then defined by

$$\begin{aligned} \widetilde{H}^1(K) = \left\{ d\nu \in \mathcal{M}_{\mathbb{C}}(\mathbb{R}) \mid \text{supp}(d\chi) \subseteq K; \int_K d\chi(x) = 0; \right. \\ \left. \|d\nu\|_{\widetilde{H}^1(K)} = \|F_+\|_{L^1(K;dx)} + \|F_-\|_{L^1(K;dx)} < \infty \right\}. \end{aligned} \quad (\text{B.44})$$

Theorem B.7 ([81], Theorem 1). *Let $K \subset \mathbb{R}$ be compact and of positive Lebesgue measure, $|K| > 0$. Then the following two items are equivalent:*

- (i) *K is homogeneous.*
- (ii) *Every element $d\nu \in \widetilde{H}^1(K)$ is of the form*

$$d\nu(x) = f(x)dx, \text{ where } f \in \widetilde{H}^1(\mathbb{R}), \text{ supp}(f) \subseteq K; \quad (\text{B.45})$$

$$\text{and hence } f \in L^1(\mathbb{R}; dx) \text{ and } \int_K dx f(x) = 0. \quad (\text{B.46})$$

In particular, $d\nu$ is purely absolutely continuous.

It is mentioned in [81] that the implication (i) implies (ii) can be inferred from previous (apparently, unpublished) work by P. W. Jones.

We return to properties of Herglotz functions.

Theorem B.8 ([29], [32]). *Let m be a Herglotz function with representation (B.3) and denote by Λ the set*

$$\Lambda = \{\lambda \in \mathbb{R} \mid \text{Im}[m(\lambda + i0)] \text{ exists (finitely or infinitely)}\}. \quad (\text{B.47})$$

Then, S , S_{ac} , S_{s} , S_{sc} , S_{pp} are essential supports of $d\omega$, $d\omega_{\text{ac}}$, $d\omega_{\text{s}}$, $d\omega_{\text{sc}}$, $d\omega_{\text{pp}}$, respectively, where

$$S = \{\lambda \in \Lambda \mid 0 < \text{Im}[m(\lambda + i0)] \leq \infty\}, \quad (\text{B.48})$$

$$S_{\text{ac}} = \{\lambda \in \Lambda \mid 0 < \text{Im}[m(\lambda + i0)] < \infty\}, \quad (\text{B.49})$$

$$S_{\text{s}} = \{\lambda \in \Lambda \mid \text{Im}[m(\lambda + i0)] = \infty\}, \quad (\text{B.50})$$

$$S_{\text{sc}} = \left\{ \lambda \in \Lambda \mid \text{Im}[m(\lambda + i0)] = \infty, \lim_{\varepsilon \downarrow 0} (-i\varepsilon)m(\lambda + i\varepsilon) = 0 \right\}, \quad (\text{B.51})$$

$$S_{\text{pp}} = \left\{ \lambda \in \Lambda \mid \text{Im}[m(\lambda + i0)] = \infty, \lim_{\varepsilon \downarrow 0} (-i\varepsilon)m(\lambda + i\varepsilon) = \omega(\{\lambda\}) > 0 \right\}. \quad (\text{B.52})$$

Moreover, since

$$|\{\lambda \in \mathbb{R} \mid |m(\lambda + i0)| = \infty, \text{Im}[m(\lambda + i0)] < \infty\}| = 0, \quad (\text{B.53})$$

$$\omega(\{\lambda \in \mathbb{R} \mid |m(\lambda + i0)| = \infty, \text{Im}[m(\lambda + i0)] < \infty\}) = 0, \quad (\text{B.54})$$

also S'_{s} , S'_{sc} , S'_{pp} are essential supports of $d\omega_{\text{s}}$, $d\omega_{\text{sc}}$, $d\omega_{\text{pp}}$, respectively, where

$$S'_{\text{s}} = \{\lambda \in \Lambda \mid |m(\lambda + i0)| = \infty\}, \quad (\text{B.55})$$

$$S'_{\text{sc}} = \left\{ \lambda \in \Lambda \mid |m(\lambda + i0)| = \infty, \lim_{\varepsilon \downarrow 0} (-i\varepsilon)m(\lambda + i\varepsilon) = 0 \right\}, \quad (\text{B.56})$$

$$S'_{\text{pp}} = \left\{ \lambda \in \Lambda \mid |m(\lambda + i0)| = \infty, \lim_{\varepsilon \downarrow 0} (-i\varepsilon)m(\lambda + i\varepsilon) = \omega(\{\lambda\}) > 0 \right\}. \quad (\text{B.57})$$

In particular,

$$S_{\text{s}} \sim S'_{\text{s}}, \quad S_{\text{sc}} \sim S'_{\text{sc}}, \quad S_{\text{pp}} \sim S'_{\text{pp}}. \quad (\text{B.58})$$

Next, consider Herglotz functions $\pm m_{\pm}$ of the type (B.3),

$$\begin{aligned} \pm m_{\pm}(z) &= c_{\pm} + d_{\pm}z + \int_{\mathbb{R}} d\omega_{\pm}(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \\ c_{\pm} &\in \mathbb{R}, \quad d_{\pm} \geq 0, \end{aligned} \quad (\text{B.59})$$

and introduce the 2×2 matrix-valued Herglotz function M

$$M(z) = (M_{j,k}(z))_{j,k=0,1}, \quad z \in \mathbb{C}_+, \quad (\text{B.60})$$

$$M(z) = \frac{1}{m_-(z) - m_+(z)} \begin{pmatrix} 1 & \frac{1}{2}[m_-(z) + m_+(z)] \\ \frac{1}{2}[m_-(z) + m_+(z)] & m_-(z)m_+(z) \end{pmatrix}, \quad z \in \mathbb{C}_+ \quad (\text{B.61})$$

$$= C + Dz + \int_{\mathbb{R}} d\Omega(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \quad (\text{B.62})$$

$$C = C^*, \quad D \geq 0$$

with $C = (C_{j,k})_{j,k=0,1}$ and $D = (D_{j,k})_{j,k=0,1}$ 2×2 matrices and $d\Omega = (d\Omega_{j,k})_{j,k=0,1}$ a 2×2 matrix-valued nonnegative measure satisfying

$$\int_{\mathbb{R}} \frac{d|\Omega_{j,k}(\lambda)|}{1 + \lambda^2} < \infty, \quad j, k = 0, 1. \quad (\text{B.63})$$

Moreover, we introduce the trace Herglotz function M^{tr}

$$M^{\text{tr}}(z) = M_{0,0}(z) + M_{1,1}(z) = \frac{1 + m_-(z)m_+(z)}{m_-(z) - m_+(z)} \quad (\text{B.64})$$

$$= a + bz + \int_{\mathbb{R}} d\Omega^{\text{tr}}(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \quad (\text{B.65})$$

$$a \in \mathbb{R}, \quad b \geq 0, \quad d\Omega^{\text{tr}} = d\Omega_{0,0} + d\Omega_{1,1}.$$

Then,

$$d\Omega \ll d\Omega^{\text{tr}} \ll d\Omega \quad (\text{B.66})$$

(where $d\mu \ll d\nu$ denotes that $d\mu$ is absolutely continuous with respect to $d\nu$).

Theorem B.9 ([30], [31]). *Let m_{\pm} , M , and M^{tr} be a Herglotz functions with representations (B.59), (B.62), and (B.65), respectively, and denote by Λ_{\pm} the sets*

$$\Lambda_{\pm} = \{\lambda \in \mathbb{R} \mid \text{Im}[m_{\pm}(\lambda + i0)] \text{ exists (finitely or infinitely)}\}. \quad (\text{B.67})$$

Then, $S_{\Omega^{\text{tr}}}$, $S_{\Omega^{\text{tr}}, \text{ac}}$, $S_{\Omega^{\text{tr}}, \text{s}}$, $S_{\Omega^{\text{tr}}, \text{sc}}$, $S_{\Omega^{\text{tr}}, \text{pp}}$ are essential supports of $d\Omega^{\text{tr}}$, $d\Omega_{\text{ac}}^{\text{tr}}$, $d\Omega_{\text{s}}^{\text{tr}}$, $d\Omega_{\text{sc}}^{\text{tr}}$, $d\Omega_{\text{pp}}^{\text{tr}}$, respectively, where

$$S_{\Omega^{\text{tr}}} = \mathbb{R} \setminus \{\lambda \in \Lambda_+ \cap \Lambda_- \mid m_{\pm}(\lambda + i0) \in \mathbb{R}, m_+(\lambda + i0) \neq m_-(\lambda + i0)\}, \quad (\text{B.68})$$

$$\begin{aligned} &\sim \{\lambda \in \Lambda_+ \cap \Lambda_- \mid 0 < \text{Im}[m_+(\lambda + i0)] < \infty\} \\ &\cup \{\lambda \in \Lambda_+ \cap \Lambda_- \mid 0 < -\text{Im}[m_-(\lambda + i0)] < \infty\} \\ &\cup \{\lambda \in \Lambda_+ \cap \Lambda_- \mid m_+(\lambda + i0) = m_-(\lambda + i0) \in \mathbb{R}\} \\ &\cup \{\lambda \in \Lambda_+ \cap \Lambda_- \mid |m_+(\lambda + i0)| = |m_-(\lambda + i0)| = \infty\}, \end{aligned} \quad (\text{B.69})$$

$$\begin{aligned} S_{\Omega^{\text{tr}}, \text{ac}} &= \{\lambda \in \Lambda_+ \cap \Lambda_- \mid 0 < \text{Im}[m_+(\lambda + i0)] < \infty\} \\ &\cup \{\lambda \in \Lambda_+ \cap \Lambda_- \mid 0 < -\text{Im}[m_-(\lambda + i0)] < \infty\}, \end{aligned} \quad (\text{B.70})$$

$$\begin{aligned} S_{\Omega^{\text{tr}}, \text{s}} &= \{\lambda \in \Lambda_+ \cap \Lambda_- \mid m_+(\lambda + i0) = m_-(\lambda + i0) \in \mathbb{R}\} \\ &\cup \{\lambda \in \Lambda_+ \cap \Lambda_- \mid |m_+(\lambda + i0)| = |m_-(\lambda + i0)| = \infty\}, \end{aligned} \quad (\text{B.71})$$

$$S_{\Omega^{\text{tr}}, \text{sc}} = \left\{ \lambda \in S_{\Omega^{\text{tr}}, \text{s}} \mid \lim_{\varepsilon \downarrow 0} (-i\varepsilon) M^{\text{tr}}(\lambda + i\varepsilon) = 0 \right\}, \quad (\text{B.72})$$

$$S_{\Omega^{\text{tr}}, \text{pp}} = \left\{ \lambda \in S_{\Omega^{\text{tr}}, \text{s}} \mid \lim_{\varepsilon \downarrow 0} (-i\varepsilon) M^{\text{tr}}(\lambda + i\varepsilon) < 0 \right\}. \quad (\text{B.73})$$

Theorem B.9 can be applied to the Schrödinger operator H defined in (2.3) assuming (for simplicity) that the associated differential expression $L = -d^2/dx^2 + V(x)$ is in the limit point case at $+\infty$ and $-\infty$ (to avoid defining separated boundary conditions at $=\infty$ and/or $-\infty$). To this end we now identify

$$m_{\pm}(z) \text{ and } m_{\pm}(z, x_0), \quad z \in \mathbb{C}_+, \quad (\text{B.74})$$

where $m_{\pm}(z, x_0)$ are the half-line Weyl–Titchmarsh m -functions associated with the Schrödinger operators H_{\pm, x_0} in $L^2([x_0, \pm\infty); dx)$ with a Dirichlet boundary condition at x_0 (cf. (2.8)).

One then obtains the following basic result.

Theorem B.10 ([31], [40], [41], [70]).

(i) *The spectral multiplicity of H is two if and only if*

$$|\mathcal{M}_2| > 0, \quad (\text{B.75})$$

where

$$\mathcal{M}_2 = \{ \lambda \in \Lambda_+ \mid m_+(\lambda + i0, x_0) \in \mathbb{C} \setminus \mathbb{R} \} \cap \{ \lambda \in \Lambda_- \mid m_-(\lambda + i0, x_0) \in \mathbb{C} \setminus \mathbb{R} \}. \quad (\text{B.76})$$

If $|\mathcal{M}_2| = 0$, the spectrum of H is simple. Moreover, \mathcal{M}_2 is a maximal set on which H has uniform multiplicity two.

(ii) *A maximal set \mathcal{M}_1 on which H has uniform multiplicity one is given by*

$$\begin{aligned} \mathcal{M}_1 = & \{ \lambda \in \Lambda_+ \cap \Lambda_- \mid m_+(\lambda + i0, x_0) = m_-(\lambda + i0, x_0) \in \mathbb{R} \} \\ & \cup \{ \lambda \in \Lambda_+ \cap \Lambda_- \mid |m_+(\lambda + i0, x_0)| = |m_-(\lambda + i0, x_0)| = \infty \} \\ & \cup \{ \lambda \in \Lambda_+ \cap \Lambda_- \mid m_+(\lambda + i0, x_0) \in \mathbb{R}, m_-(\lambda + i0, x_0) \in \mathbb{C} \setminus \mathbb{R} \} \\ & \cup \{ \lambda \in \Lambda_+ \cap \Lambda_- \mid m_-(\lambda + i0, x_0) \in \mathbb{R}, m_+(\lambda + i0, x_0) \in \mathbb{C} \setminus \mathbb{R} \}. \end{aligned} \quad (\text{B.77})$$

In particular, $\sigma_s(H) = \sigma_{\text{sc}}(H) \cup \sigma_{\text{pp}}(H)$ is always simple.

Acknowledgments. We are indebted to Konstantin Makarov, Franz Peherstorfer, Misha Sodin, and Michel Zinsmeister for helpful discussions, and especially thank Konstantin Makarov for his help in connection with the material in Appendices A and B.

Fritz Gesztesy gratefully acknowledges the extraordinary hospitality of Helge Holden and the Department of Mathematical Sciences of the Norwegian University of Science and Technology, Trondheim, during two two-month stays in the summers of 2004 and 2005, where parts of this paper were written. He also gratefully acknowledges a research leave for the academic year 2005/06 granted by the Office of Research of the University of Missouri–Columbia.

Peter Yuditskii wishes to thank Franz Peherstorfer and the Section for Dynamical Systems and Approximation Theory, University of Linz, Austria, for the great hospitality extended to him from September 2003 to September 2005 and he is particularly grateful to Franz Peherstorfer for creating a stimulating atmosphere and numerous discussions on the topic at hand. He also thanks Stas Kupin and the Department of Mathematics of CMI, Université de Provence, Marseille, France, for their kind hospitality during a one-month stay in the summer of 2004, where a part of this paper was written.

REFERENCES

- [1] N. I. Akhiezer and I. M. Glazman, *Theory of Operators in Hilbert Space*, Vol. I, Pitman, Boston, 1981.
- [2] N. Aronszajn, *On a problem of Weyl in the theory of singular Sturm–Liouville equations*, Amer. J. Math. **79**, 597–610 (1957).
- [3] N. Aronszajn and W. F. Donoghue, *On exponential representations of analytic functions in the upper half-plane with positive imaginary part*, J. Analyse Math. **5**, 321–388 (1956–57).
- [4] N. Aronszajn and W. F. Donoghue, *A supplement to the paper on exponential representations of analytic functions in the upper half-plane with positive imaginary parts*, J. Analyse Math. **12**, 113–127 (1964).
- [5] J. Avron and B. Simon, *Almost periodic Schrödinger operators I. Limit periodic potentials*, Commun. Math. Phys. **82**, 101–120 (1981).
- [6] E. D. Belokolos, A. I. Bobenko, V. Z. Enol’skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.
- [7] D. Buschmann, *Spektraltheorie verallgemeinerter Differentialausdrücke – Ein neuer Zugang*, Ph.D. Thesis, University of Frankfurt, Germany, 1997.
- [8] C. Caratheodory, *Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten*, Math. Ann. **72**, 107–144 (1912).
- [9] L. Carleson, *On H^∞ in multiply connected domains*, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, Vol. II, W. Beckner, A. P. Calderón, R. Fefferman, and P. W. Jones (eds.), Wadsworth, CA, 1983, pp. 349–372.
- [10] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Basel, 1990.
- [11] V. A. Chulaevskii, *Inverse spectral problem for limit-periodic Schrödinger operators*, Funct. Anal. Appl. **18**, 230–233 (1984).
- [12] S. Clark and F. Gesztesy, *Weyl–Titchmarsh M -function asymptotics for matrix-valued Schrödinger operators*, Proc. London Math. Soc. **82**, 701–724 (2001).
- [13] S. Clark and F. Gesztesy, *On Povzner–Wienholtz-type self-adjointness results for matrix-valued Sturm–Liouville operators*, Proc. Roy. Soc. Edinburg **133A**, 747–758 (2003).
- [14] S. Clark, F. Gesztesy, H. Holden, and B. M. Levitan, *Borg-type theorems for matrix-valued Schrödinger operators*, J. Diff. Eqs. **167**, 181–210 (2000).
- [15] W. Craig, *The trace formula for Schrödinger operators on the line*, Commun. Math. Phys. **126**, 379–407 (1989).
- [16] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators*, Springer, Berlin, 1987.
- [17] P. Deift and B. Simon, *Almost periodic Schrödinger operators III. The absolutely continuous spectrum in one dimension*, Commun. Math. Phys. **90**, 389–411 (1983).
- [18] R. del Rio, B. Simon, and G. Stolz, *Stability of spectral types for Sturm–Liouville operators*, Math. Res. Lett. **1**, 437–450 (1994).
- [19] W. F. Donoghue, *Monotone Matrix Functions and Analytic Continuation*, Springer, Berlin, 1974.
- [20] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, *Non-linear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties*, Russian Math. Surv. **31:1**, 59–146 (1976).
- [21] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
- [22] I. E. Egorova, *On a class of almost periodic solutions of the KdV equation with a nowhere dense spectrum*, Russian Acad. Sci. Dokl. Math. **45**, 290–293 (1990).
- [23] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
- [24] I. M. Gelfand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Izv. Akad. Nauk SSR. Ser. Mat. **15**, 309–360 (1951) (Russian); English transl. in Amer. Math. Soc. Transl. Ser. 2 **1**, 253–304 (1955).
- [25] F. Gesztesy, *A complete spectral characterization of the double commutation method*, J. Funct. Anal. **117**, 401–446 (1993).
- [26] F. Gesztesy and H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions. Vol. I: $(1+1)$ -Dimensional Continuous Models*, Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge Univ. Press, 2003.

- [27] F. Gesztesy, M. Krishna, and G. Teschl, *On isospectral sets of Jacobi operators*, Commun. Math. Phys. **181**, 631–645 (1996).
- [28] F. Gesztesy and B. Simon, *The ξ function*, Acta Math. **176**, 49–71 (1996).
- [29] D. J. Gilbert, *Subordinacy and Spectral Analysis of Schrödinger Operators*, Ph.D. Thesis, University of Hull, 1984.
- [30] D. J. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints*, Proc. Roy. Soc. Edinburgh **112A**, 213–229 (1989).
- [31] D. J. Gilbert, *On subordinacy and spectral multiplicity for a class of singular differential operators*, Proc. Roy. Soc. Edinburgh **A 128**, 549–584 (1998).
- [32] D. J. Gilbert and D. B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128**, 30–56 (1987).
- [33] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Amer. Math. Soc., Providence, RI, 1969.
- [34] P. Hartman, *Differential equations with non-oscillatory eigenfunctions*, Duke Math. J. **15**, 697–709 (1948).
- [35] M. Hasumi, *Hardy Classes of Infinitely Connected Riemann Surfaces*, Lecture Notes in Math. **1027**, Springer, Berlin, 1983.
- [36] R. A. Johnson, *The recurrent Hill's equation*, J. Diff. Eqs. **46**, 165–193 (1982).
- [37] R. A. Johnson, *On the Sato–Segal–Wilson solutions of the K – dV equation*, Pac. J. Math. **132**, 343–3355 (1988).
- [38] R. Johnson and J. Moser, *The rotation number for almost periodic potentials*, Commun. Math. Phys. **84**, 403–438 (1982).
- [39] P. W. Jones and D. E. Marshall, *Critical points of Green's function, harmonic measure, and the corona problem*, Ark. Mat. **23**, 281–314 (1985).
- [40] I. S. Kac, *On the multiplicity of the spectrum of a second-order differential operator*, Sov. Math. Dokl. **3**, 1035–1039 (1962).
- [41] I. S. Kac, *Spectral multiplicity of a second order differential operator and expansion in eigenfunctions*, Izv. Akad. Nauk SSSR **27**, 1081–1112 (1963). Erratum, Izv. Akad. Nauk SSSR **28**, 951–952 (1964). (Russian.)
- [42] I. S. Kac and M. G. Krein, *R -functions—analytic functions mapping the upper halfplane into itself*, Amer. Math. Soc. Transl. (2) **103**, 1–18 (1974).
- [43] P. Koosis, *Introduction to H_p Spaces*, 2nd ed., Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1998.
- [44] S. Kotani, *Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators*, in “Stochastic Analysis”, K. Itô (ed.), North-Holland, Amsterdam, 1984, pp. 225–247.
- [45] S. Kotani, *One-dimensional random Schrödinger operators and Herglotz functions*, in “Probabilistic Methods in Mathematical Physics”, K. Itô and N. Ikeda (eds.), Academic Press, New York, 1987, pp. 219–250.
- [46] S. Kotani, *Link between periodic potentials and random potentials in one-dimensional Schrödinger operators*, in *Differential Equations and Mathematical Physics*, I. W. Knowles (ed.), Springer, Berlin, 1987, pp. 256–269.
- [47] S. Kotani and M. Krishna, *Almost periodicity of some random potentials*, J. Funct. Anal. **78**, 390–405 (1988).
- [48] B. M. Levitan, *Almost periodicity of infinite-zone potentials*, Math. USSR Izvestija **18**, 249–273 (1982).
- [49] B. M. Levitan, *Approximation of infinite-zone potentials by finite-zone potentials*, Math USSR Izvestija **20**, 55–87 (1983).
- [50] B. M. Levitan, *An inverse problem for the Sturm–Liouville operator in the case of finite-zone and infinite-zone potentials*, Trans. Moscow Math. Soc. **45:1**, 1–34 (1984).
- [51] B. M. Levitan, *On the closure of the set of finite-zone potentials*, Math. USSR Sbornik **51**, 67–89 (1985).
- [52] B. M. Levitan, *Inverse Sturm–Liouville Problems*, VNU Science Press, Utrecht, 1987.
- [53] B. M. Levitan and M. G. Gasymov, *Determination of a differential equation by two of its spectra*, Russ. Math. Surv. **19:2**, 1–63 (1964).
- [54] B. M. Levitan and A. V. Savin, *Examples of Schrödinger operators with almost periodic potentials and nowhere dense absolutely continuous spectrum*, Sov. Math. Dokl. **29**, 541–544 (1984).

- [55] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.
- [56] J. Moser, *An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum*, Comment. Math. Helvetici **56**, 198–224 (1981).
- [57] M. A. Naimark, *Linear Differential Operators, Part II*, F. Ungar, New York, 1968.
- [58] S. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York, 1984.
- [59] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer, Berlin, 1992.
- [60] L. A. Pastur and V. A. Tkachenko, *Spectral theory of a class of one-dimensional Schrödinger operators with limit-periodic potentials*, Trans. Moscow Math. Soc. 1989, 115–166.
- [61] F. Peherstorfer and P. Yuditskii, *Asymptotic behavior of polynomials orthonormal on a homogeneous set*, J. Analyse Math. **89**, 113–154 (2003).
- [62] I. I. Priwalow, *Randeigenschaften analytischer Funktionen*, 2nd ed., VEB Verlag, Berlin, 1956.
- [63] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV: Analysis of Operators*, Academic Press, New York, 1978.
- [64] F. Rellich, *Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung*, Math. Ann. **122**, 343–368 (1951).
- [65] F. S. Roze-Beketov, *Expansions in eigenfunctions of infinite systems of differential equations in the non-self-adjoint and self-adjoint cases*, Mat. Sb. **51**, 293–342 (1960). (Russian.)
- [66] M. Rosenblum and J. Rovnyak, *Topics in Hardy Classes and Univalent Functions*, Birkhäuser, Basel, 1994.
- [67] A. Rybkin, *On the trace approach to the inverse scattering problem in dimension one*, SIAM J. Math. Anal. **32**, 1248–1264 (2001).
- [68] B. Simon, *Spectral analysis of rank one perturbations and applications*, CRM Proceedings and Lecture Notes **8**, 109–149 (1995).
- [69] B. Simon, *L^p norms of the Borel transform and the decomposition of measures*, Proc. Amer. Math. Soc. **123**, 3749–3755 (1995).
- [70] B. Simon, *On a theorem of Kac and Gilbert*, J. Funct. Anal. **223**, 109–115 (2005).
- [71] B. Simon and T. Wolff, *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*, Commun. Pure Appl. Math. **39**, 75–90 (1986).
- [72] M. Sodin and P. Yuditskii, *Almost periodic Sturm–Liouville operators with Cantor homogeneous spectrum and pseudoextendible Weyl functions*, Russ. Acad. Sci. Dokl. Math. **50**, 512–515 (1995).
- [73] M. Sodin and P. Yuditskii, *Almost periodic Sturm–Liouville operators with Cantor homogeneous spectrum*, Comment. Math. Helvetici **70**, 639–658 (1995).
- [74] M. Sodin and P. Yuditskii, *Almost-periodic Sturm–Liouville operators with homogeneous spectrum*, in *Algebraic and Geometric Methods in Mathematical Physics*, A. Boutet de Monvel and A. Marchenko (eds.), Kluwer, 1996, pp. 455–462.
- [75] M. Sodin and P. Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, J. Geom. Anal. **7**, 387–435 (1997).
- [76] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.
- [77] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. Monographs, Vol. 72, Amer. Math. Soc., Providence, R.I., 2000.
- [78] C. Thurlow, *A generalisation of the inverse spectral theorem of Levitan and Gasymov*, Proc. Roy. Soc. Edinburgh **84A**, 185–196 (1979).
- [79] H. Widom, *The maximum principle for multiple-valued analytic functions*, Acta Math. **126**, 63–82 (1971).
- [80] H. Widom, *H^p sections of vector bundles over Riemann surfaces*, Ann. Math. **94**, 304–324 (1971).
- [81] M. Zinsmeister, *Espaces de Hardy et domaines de Denjoy*, Ark. Mat. **27**, 363–378 (1989).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA
 E-mail address: fritz@math.missouri.edu
 URL: <http://www.math.missouri.edu/personnel/faculty/gesztesyf.html>

DEPARTMENT OF MATHEMATICS, BAR ILAN UNIVERSITY, 52900 RAMAT GAN, ISRAEL
E-mail address: `yuditski@macs.biu.ac.il`